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Large deviations and renormalization for Riesz potentials of stable intersection measures

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Abstract

We study the object formally defined as

$$\gamma\left([0,t]^2\right) = \iint_{[0,t]^2} |X_s - X_r|^{-\sigma}\, \mathrm{d}r\, \mathrm{d}s - E \iint_{[0,t]^2} |X_s - X_r|^{-\sigma}\, \mathrm{d}r\, \mathrm{d}s, \tag{0.1}$$

where X_t denotes the symmetric stable processes of index $0 < \beta \leq 2$ in R^d. When $\beta \leq \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$, this has to be defined as a limit, in the spirit of renormalized self-intersection local time. We obtain results about the large deviations and laws of the iterated logarithm for γ. This is applied to obtain results about stable processes in random potentials.

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1. Introduction

Let X_t be a d-dimensional symmetric stable process of index $0 < \beta \leq 2$. Thus we assume that there is a continuous function $\psi(\lambda)$ on $\mathbb{R}^d$ which is strictly positive for $|\lambda| \neq 0$, with

$$\psi(r\lambda) = r^{\beta}\psi(\lambda) \quad \text{and} \quad \psi(-\lambda) = \psi(\lambda), \quad r > 0, \lambda \in \mathbb{R}^d$$

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such that

$$\mathbb{E}e^{i\lambda\cdot X_t} = e^{-t\psi(\lambda)}, \quad t \geq 0, \lambda \in \mathbb{R}^d. \tag{1.1}$$

It follows that there is a constant $C > 0$ such that

$$C^{-1}|\lambda|^\beta \leq \psi(\lambda) \leq C|\lambda|^\beta, \quad \lambda \in \mathbb{R}^d.$$

In this paper we study

$$\eta(A) = \iint_A |X_s - X_r|^{-\sigma}\, dr\, ds, \quad A \subset R_+^2 \tag{1.2}$$

and, more generally,

$$\eta^z(A) = \iint_A |X_s - X_r - z|^{-\sigma}\, dr\, ds \tag{1.3}$$

for $z \in \mathbb{R}^d$. We are particularly interested in the case $A = [0, t]^2$ or $[0, t]_<^2$, where for any $t > 0$,

$$[0, t]_<^2 = \left\{(r, s) \in [0, t]^2; r < s\right\}.$$

Thus we will study

$$\eta^z([0, t]_<^2) = \iint_{[0,t]_<^2} |X_s - X_r - z|^{-\sigma}\, dr\, ds. \tag{1.4}$$

We can write

$$\eta^z([0, t]_<^2) = \int_{\mathbb{R}^d} \frac{1}{|x - z|^\sigma}\, \mu_{[0,t]_<^2}(dx), \tag{1.5}$$

where μ_A for $A \subseteq \mathbb{R}_+^2$ is the measure on $\mathbb{R}^d$ defined by

$$\mu_A(B) = \iint_A 1_{\{X_s - X_r \in B\}}\, dr\, ds. \tag{1.6}$$

We refer to μ_A as the intersection measure for the stable process X_t, since whenever $\mu_{[0,t]_<^2}$ has a density $\alpha_t(x)$ which is continuous at x, $\alpha_t(x)$ is the intersection local time for X_t. In particular, if $\alpha_t(x)$ is continuous at $x = 0$, $\alpha_t(0)$ is a 'measure' of the set $\{(r, s) \in [0, t]_<^2 \mid X_s = X_r\}$.

(1.5) shows that $\eta^z([0, t]_<^2)$ is the Riesz potential of the intersection measure $\mu_{[0,t]_<^2}$. (In the terminology of [12], $\eta^z([0, t]_<^2)$ is the Riesz–Frostman potential of the intersection measure.)

Notice that $\eta([0, t]^2) = 2\eta([0, t]_<^2)$.

When $0 < \sigma < \min\{\beta, d\}$,

$$\mathbb{E}\iint_{[0,t]_<^2} |X_s - X_r|^{-\sigma}\, dr\, ds = \mathbb{E}(|X_1|^{-\sigma}) \int_0^t \int_0^s \frac{1}{(s - r)^{\sigma/\beta}}\, dr\, ds \tag{1.7}$$

is finite for all $t \geq 0$, so that $\eta\left([0, t]_<^2\right) < \infty$, a.s.

We are interested in Riesz potentials of intersection measures for two reasons. First, our investigation is motivated by applications to polymer models. Mathematically, a random polymer is modeled as a random path ω whose probability measure is given in terms of the Gibbs measure

$$P_t(\omega) = \frac{1}{Z_t} e^{\pm H_t(\omega)}\, d\omega. \tag{1.8}$$

Here $H_t(\omega) \geq 0$ is a suitable Hamiltonian which describes the interaction between the monomers along the path $\omega = \{X_s; 0 \leq s \leq t\}$, $\mathrm{d}\omega$ represents the underlying measure on path space, and $Z_t = E\left(\mathrm{e}^{\pm H_t(\omega)}\right)$ is the normalization. In most models, the role of H_t is to reward or penalize attraction between monomers. The first case, with $+H_t$, describes a "self-attracting" polymer, while the second case, with $-H_t$, describes a "self-repelling" polymer. We refer to the recent book by den Hollander, [18], for a systematic overview of polymer models.

In the existing literature, H_t is often taken to be the self-intersection local time, formally defined as

$$H_t = \int_0^t \int_0^t \delta_0(X_r - X_s)\mathrm{d}r\mathrm{d}s. \tag{1.9}$$

In this model, the monomers along the path interact only when they intersect.

If one believes that all monomers along the path interact, but the strength of the interaction decreases with distance, then the choice of

$$H_t = \int_0^t \int_0^t |X_r - X_s|^{-\sigma}\mathrm{d}r\mathrm{d}s \tag{1.10}$$

would be a more realistic model.

The second reason for our interest in Riesz potentials of intersection measures arises from the "polaron problem", which originated in electrostatics. See [17,26] for general information. The integral in (1.10) is associated with the asymptotics of the mean-field, or long range interaction, polaron, while the integral

$$\int_0^t \int_0^t \frac{\mathrm{e}^{-|r-s|}}{|X_r - X_s|^{\sigma}}\mathrm{d}r\mathrm{d}s$$

is associated with a polaron with interactions which are exponentially damped in time. Donsker and Varadhan [14] solved a long standing problem in physics by showing that, for Brownian motion W_t in R^3,

$$\mathcal{D}(\theta) \equiv \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{\theta \int_0^t \int_0^t \frac{\mathrm{e}^{-|r-s|}}{|W_r - W_s|}\mathrm{d}r\mathrm{d}s\right\} \tag{1.11}$$

exists and

$$\lim_{\theta\to\infty} \frac{\mathcal{D}(\theta)}{\theta^2} = \sup_{g\in\mathcal{F}_2} \left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{g^2(x)g^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2\mathrm{d}x\right\}$$

where (with $d = 3$ in Donsker and Varadhan's setting)

$$\mathcal{F}_2 = \left\{g \in \mathcal{L}^2(\mathbb{R}^d);\ \|g\|_2 = 1 \text{ and } \|\nabla g\|_2 < \infty\right\}.$$

Mansmann [25] showed that for Brownian motion W_t in R^d, $d \geq 3$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{\frac{1}{t}\int_0^t \int_0^t \frac{1}{|W_s - W_r|}\,\mathrm{d}r\,\mathrm{d}s\right\}$$
$$= \sup_{g\in\mathcal{F}_2} \left\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|}\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\|\nabla f\|_2^2\right\}. \tag{1.12}$$

The results mentioned above are linked conceptually to the well known work of Donsker and Varadhan on large deviation for general Markov processes. To illustrate, we take (1.12) as an example. Viewing the empirical measure $\mathcal{L}_t(\cdot)$

$$\mathcal{L}_t(A) = \frac{1}{t}\int_0^t 1_{\{W_s \in A\}} \mathrm{d}s \quad A \in \mathcal{B}(\mathbb{R}^d)$$

as a stochastic process taking values in $\mathcal{P}(\mathbb{R}^d)$, the space of the probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence, Donsker and Varadhan [13] established a weak form of the large deviations for $\mathcal{L}_t(\cdot)$ with the rate function $I(\mu)$, where

$$I(\mu) = \frac{1}{8}\int_{\mathbb{R}^d} \frac{|\nabla f(x)|^2}{f(x)} \mathrm{d}x$$

if μ is a probability measure with density f for which the right-hand side makes sense; and $I(\mu) = \infty$ otherwise.

Define the function Ψ on (a subset of) $\mathcal{P}(\mathbb{R}^d)$ as

$$\Psi(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|}\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

By Varadhan's integral lemma (Theorem 4.3.1 in [11]), we therefore expect that

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\exp\left\{\frac{1}{t}\int_0^t\int_0^t \frac{1}{|W_s - W_r|}\,\mathrm{d}r\,\mathrm{d}s\right\} \\ &= \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\{t\,\Psi(\mathcal{L}_t)\} = \sup_{\mu\in\mathcal{P}(\mathbb{R}^d)}\{\Psi(\mu) - I(\mu)\} \\ &= \sup_f \left\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f(x)f(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{8}\int_{\mathbb{R}^d} \frac{|\nabla f(x)|^2}{f(x)}\mathrm{d}x\right\}, \end{aligned} \tag{1.13}$$

where the supremum on the right-hand side is taking for the probability density functions f on $\mathbb{R}^d$. This becomes (1.12) under the substitution $f(x) = g^2(x)$.

Turning this into a rigorous proof is highly non-trivial. The main reason is that, in Mansmann's setting, the state space $\mathbb{R}^d$ is not compact and the functional $\Psi(\cdot)$ is not continuous. Because of these difficulties, Mansmann [25] admits that his approach cannot be extended to dimensions $d = 1, 2$.

Our first main theorem is the large deviation principle for $\eta\left([0,t]^2_<\right)$. For $0 < \sigma < d$ let

$$\varphi_{d-\sigma}(\lambda) = \frac{C_{d,\sigma}}{|\lambda|^{d-\sigma}} \tag{1.14}$$

where $C_{d,\sigma} = \pi^{-d/2}2^{-\sigma}\Gamma(\frac{d-\sigma}{2})/\Gamma(\frac{\sigma}{2})$. Write

$$\rho = \sup_{||f||_2=1}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} \frac{f(\lambda+\gamma)f(\gamma)}{\sqrt{1+\psi(\lambda+\gamma)}\sqrt{1+\psi(\gamma)}}\mathrm{d}\gamma\right]^2 \varphi_{d-\sigma}(\lambda)\mathrm{d}\lambda. \tag{1.15}$$

Clearly, $\rho > 0$. According to Lemma 1.6 in [2], $0 < \rho < \infty$ when $0 < \sigma < \min\{2\beta, d\}$.

By the scaling property

$$\eta\left([0,t]^2_<\right) \stackrel{d}{=} t^{2-\beta^{-1}\sigma}\eta\left([0,1]^2_<\right), \quad t \geq 0, \tag{1.16}$$

we need only consider $\eta\left([0,1]^2_<\right)$ in the following theorem.

Theorem 1.1. *When* $0 < \sigma < \min\{\beta, d\}$,

$$\lim_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{\eta\left([0,1]^2_<\right) \ge a\right\} = -(2\rho)^{-\beta/\sigma} \frac{\sigma}{\beta} \left(\frac{2\beta-\sigma}{\beta}\right)^{\frac{2\beta-\sigma}{\sigma}}. \tag{1.17}$$

Using variation relations established in [2], Theorem 1.1 can be shown to agree with the heuristic formula (1.13), suitably modified for stable processes.

Using scaling, (1.16), the relation $\eta([0,1]) = 2\eta\left([0,1]^2_<\right)$ and Varadhan's integral lemma, [11, Section 4.3], we obtain the asymptotics

$$\lim_{t\to\infty} t^{-\frac{2\beta-\sigma}{\beta-\sigma}} \log \mathbb{E} \exp \left\{\int_0^t \int_0^t |X_r - X_s|^{-\sigma} \mathrm{d}r \mathrm{d}s\right\}$$
$$= 2^{\frac{2\beta}{\beta-\sigma}} \frac{\beta-\sigma}{\beta} \left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{\beta-\sigma}} \rho^{\frac{\beta}{\beta-\sigma}} \tag{1.18}$$

for the partition function in the self-attracting polymer model with Hamiltonian defined in (1.10). Note that when $d = 3$, $\beta = 2$ and $\sigma = 1$ this becomes

$$\lim_{t\to\infty} t^{-3} \log \mathbb{E} \exp \left\{\int_0^t \int_0^t \frac{1}{|W_r - W_s|} \mathrm{d}r\, \mathrm{d}s\right\} = \frac{2^6}{3^3}\rho^2. \tag{1.19}$$

Comparison with (1.11) shows a striking difference between the asymptotics of polarons with long range interactions and those with exponentially damped interactions.

Theorem 1.1 implies the following laws of the iterated logarithm for $\eta\left([0,t]^2_<\right)$.

Theorem 1.2. *When* $0 < \sigma < \min\{\beta, d\}$,

$$\limsup_{t\to\infty} t^{-\frac{2\beta-\sigma}{\beta}} (\log\log t)^{-\sigma/\beta} \eta\left([0,t]^2_<\right) = 2\rho \left(\frac{\beta}{\sigma}\right)^{\sigma/\beta} \left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{\beta}}, \tag{1.20}$$

almost surely.

We are also interested in the situation where $\beta \le \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$. In this case $\mathbb{E}\eta\left([0,t]^2_<\right) = \infty$ by (1.7). We intend to show how to make sense of the object formally given by

$$\iint_{\{0\le r<s\le t\}} |X_s - X_r|^{-\sigma} \mathrm{d}r \mathrm{d}s - \mathbb{E} \iint_{\{0\le r<s\le t\}} |X_s - X_r|^{-\sigma} \mathrm{d}r \mathrm{d}s. \tag{1.21}$$

This is reminiscent of the situation for Brownian intersection local time in R^2. In that case the measure $\mu_{[0,t]^2_<}$ defined in (1.6) has a density $\alpha_t(x)$ which is continuous for all $x \neq 0$, but not for $x = 0$. To make sense of $\alpha_t(0)$ we must 'renormalize'. This was first done by Varadhan [32], and has been the subject of a large literature, see [15,23,3,28]. The resulting renormalized intersection local time turns out to be the right tool for the solution of certain "classical" problems such as the asymptotic expansion of the area of the Wiener and stable sausage in the plane and fluctuations of the range of stable random walks. (See [22,21,24,27].)

There are now several ways to 'renormalize' the Brownian intersection local time $\alpha_t(0)$. We briefly recall one such method, since we will use a similar method to renormalize $\eta\left([0,t]^2_<\right)$. Let us write $\alpha(x, A)$ for the density of the measure μ_A. Thus, $\alpha_t(x) = \alpha(x, [0, t]^2_<)$. It can be shown that for any $a < b \le c < d$, $\mu_{[a,b]\times[c,d]}$ has a continuous density $\alpha(x, [a, b] \times [c, d])$. We then note that $[0, t]^2_<$ has a decomposition as

$$[0, t]^2_< = \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{2^k-1} A_l^k \tag{1.22}$$

where

$$A_l^k = \left[\frac{2l}{2^{k+1}}t, \frac{2l+1}{2^{k+1}}t\right) \times \left[\frac{2l+1}{2^{k+1}}t, \frac{2l+2}{2^{k+1}}t\right). \tag{1.23}$$

It can then be shown that

$$\sum_{k=0}^{\infty} \sum_{l=0}^{2^k-1} \left(\alpha(0, A_l^k) - E(\alpha(0, A_l^k))\right) \tag{1.24}$$

converges, and the limit is called renormalized intersection local time.

In the following section we shall carry out a similar program to renormalize $\eta\left([0, t]^2_<\right)$, which will give meaning to the formal expression in (1.21). The resulting object will be denoted by $\gamma\left([0, t]^2_<\right)$. This will make perfectly good sense in the context of (1.8), which will now be "renormalized" to

$$P_t(\omega) = \frac{1}{\widetilde{Z}_t} \mathrm{e}^{\pm\gamma([0,t]^2)} \,\mathrm{d}\omega. \tag{1.25}$$

The second main result of this paper is to show that $\gamma\left([0, t]^2_<\right)$ has large deviation properties and laws of the iterated logarithm similar to those established above for $\eta\left([0, t]^2_<\right)$ when $0 < \sigma < \min\{\beta, d\}$.

Theorem 1.3. *When* $\beta \le \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$,

$$\lim_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{\gamma\left([0, 1]^2_<\right) \ge a\right\} = -2^{-\beta/\sigma} \frac{\sigma}{\beta} \left(\frac{2\beta - \sigma}{\beta}\right)^{\frac{2\beta-\sigma}{\sigma}} \rho^{-\beta/\sigma}. \tag{1.26}$$

Consider the special case $\beta = \sigma < d$. Combined with the scaling property given in (2.9) below, Theorem 1.3 shows that the self-attracting polymer measure (1.25) 'collapses' in finite time, by which we mean that

$$\mathbb{E}\exp\left\{\gamma\left([0, t]^2\right)\right\} \begin{cases} <\infty & t < \rho^{-1} \\ =\infty & t > \rho^{-1}. \end{cases} \tag{1.27}$$

Theorem 1.3 also indicates that the self-attracting polymer with $H_t = \gamma([0, t]^2_<)$ is not well defined when $\sigma > \beta$. But it is not hard to show that

$$\mathbb{E}\exp\left\{-\gamma([0, t]^2_<)\right\} < \infty$$

for every $t > 0$ if $\beta \le \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$. A problem relevant to the self-repelling polymer is to investigate the asymptotics of

$$\mathbb{E}\exp\left\{-\eta\left([0,t]^2_<\right)\right\} \quad \text{or} \quad \mathbb{E}\exp\left\{-\gamma\left([0,t]^2_<\right)\right\}$$

as $t \to \infty$. We leave this to future study.

Theorem 1.3 implies the following laws of the iterated logarithm for $\gamma\left([0,t]^2_<\right)$.

Theorem 1.4. *When $\beta \le \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$,*

$$\limsup_{t\to\infty} t^{-\frac{2\beta-\sigma}{\beta}}(\log\log t)^{-\sigma/\beta}\gamma\left([0,t]^2_<\right) = 2\rho\left(\frac{\beta}{\sigma}\right)^{\sigma/\beta}\left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{\beta}}, \tag{1.28}$$

almost surely.

We next describe an application to the study of stable processes in a Brownian potential. Let W denote white noise on $L^2(R^d, \mathrm{d}x)$. That is, for every $f \in L^2(R^d, \mathrm{d}x)$, $W(f)$ is the mean zero Gaussian process with covariance

$$E(W(f)W(g)) = \int_{R^d} f(x)g(x)\,\mathrm{d}x, \tag{1.29}$$

which we take to be independent of our stable process X_t. $W(f)$ can be considered as a stochastic integral

$$W(f) = \int_{R^d} f(x)\,W(\mathrm{d}x) \tag{1.30}$$

with respect to the Brownian sheet, [33].

Recall the identity

$$\int_{\mathbb{R}^d} |x-z|^{-\frac{\sigma+d}{2}}|y-z|^{-\frac{\sigma+d}{2}}\mathrm{d}z = C\frac{1}{|x-y|^{\sigma}}, \quad x, y \in \mathbb{R}^d \tag{1.31}$$

where

$$C = \pi^{d/2}\frac{\Gamma^2\left(\frac{d-\sigma}{4}\right)\Gamma\left(\frac{\sigma}{2}\right)}{\Gamma^2\left(\frac{d+\sigma}{4}\right)\Gamma\left(\frac{d-\sigma}{2}\right)}, \tag{1.32}$$

see [12, p. 118, 158] or [30, p. 118. (8)]. Then, if we set

$$\xi(t,x) = \int_0^t |X_s - x|^{-\frac{\sigma+d}{2}}\,\mathrm{d}s, \tag{1.33}$$

we see by Fubini's theorem that

$$\eta\left([0,t]^2\right) = C^{-1}\int_{\mathbb{R}^d}\xi(t,x)^2\mathrm{d}x < \infty, \quad \text{a.s.} \tag{1.34}$$

Hence, almost surely with respect to X,

$$F(t) \equiv W(\xi(t,\cdot)) = \int_{\mathbb{R}^d}\xi(t,x)W(\mathrm{d}x) \tag{1.35}$$

is a mean zero normal random variable with

$$E\left(F^2(t)\right) = (2C)\eta\left([0,t]^2_<\right). \tag{1.36}$$

By a stable process in a Brownian potential we mean the process described by the measure

$$Q_t = \frac{1}{N_t} \mathrm{e}^{-\int_0^t V(X_s)\mathrm{d}s} P_t, \tag{1.37}$$

where P_t is the probability for the stable process $\{X_s; 0 \le s \le t\}$, Z_t is the normalization and the potential function $V(x)$ is of the form

$$V(x) = \int_{\mathbb{R}^d} K(y-x)W(\mathrm{d}y), \tag{1.38}$$

where $K(x)$ is a function on $\mathbb{R}^d$ known as the shape function. Roughly speaking, (1.38) represents the interaction with a field generated by a cloud of electrons $W(\mathrm{d}y)$ with random signed charges and locations in $\mathbb{R}^d$. We refer [6,31] for more information.

When $K(x) = \delta_0(x)$, the action $\int_0^t V(X_s)\mathrm{d}s$ corresponds to a stable process in Brownian scenery [19,20,7,4,9]. When $K(x)$ is bounded and locally supported, the long term behavior of $\int_0^t V(X_s)\mathrm{d}s$ is similar to that of the stable process in Brownian scenery.

Let $\frac{d}{2} < p < \min\left\{d, \frac{d+\beta}{2}\right\}$. The random potential function

$$V_p(x) \equiv \int_{\mathbb{R}^d} \frac{1}{|y-x|^p} W(\mathrm{d}y), \quad x \in \mathbb{R}^d \tag{1.39}$$

has a very intuitive physical meaning. When $d = 3$ and $p = 1$, it represents the electrostatic potential energy generated by a cloud of electrons $W(\mathrm{d}y)$. Unfortunately, the random potential function (1.39) is not well defined, since $\mathbb{E}V_p^2(x) = \infty$ for every $x \in \mathbb{R}^d$. However, as we have seen,

$$F(t) \equiv \int_{\mathbb{R}^d} \left[\int_0^t \frac{\mathrm{d}s}{|y-X_s|^p}\right] W(\mathrm{d}y) \tag{1.40}$$

is well defined. Here we have taken $\sigma = 2p-d$. Because of (1.37), we refer to $F(t)$ as the action for a stable process in a Brownian potential.

It is easy to see that for each $t > 0$,

$$F(t) \stackrel{d}{=} t^{\frac{2\beta-2p+d}{2\beta}} F(1). \tag{1.41}$$

The following corollaries about large deviations and laws of the iterated logarithm for $F(t)$ will follow from Theorem 1.1.

Corollary 1.5. *If $\frac{d}{2} < p < \min\left\{d, \frac{d+\beta}{2}\right\}$, then*

$$\lim_{a\to\infty} a^{-\frac{2\beta}{2p-d+\beta}} \log \mathbb{P}\{\pm F(1) \ge a\}$$
$$= -\frac{\beta+2p-d}{\beta} (8C_p\rho_p)^{-\frac{\beta}{\beta+2p-d}} \left(\frac{2\beta-2p+d}{\beta}\right)^{\frac{2\beta-2p-d}{\beta+2p-d}} \tag{1.42}$$

where the constants

$$C_p = \pi^{d/2} \frac{\Gamma^2\left(\frac{d-p}{2}\right)\Gamma\left(\frac{2p-d}{2}\right)}{\Gamma^2\left(\frac{p}{2}\right)\Gamma(d-p)},$$

and

$$\rho_p = \sup_{||f||_2=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(\lambda+\gamma)f(\gamma)}{\sqrt{1+\psi(\lambda+\gamma)}\sqrt{1+\psi(\gamma)}}\,\mathrm{d}\gamma\right]^2 \varphi_{2(d-p)}(\lambda)\,\mathrm{d}\lambda$$

come from C and ρ defined in (1.32) and (1.15), respectively, with $\sigma = 2p-d$.

Corollary 1.6.

$$\limsup_{t\to\infty} t^{-\frac{2\beta-2p+d}{2\beta}}(\log\log t)^{-\frac{2p-d+\beta}{2\beta}}\{\pm F(t)\}$$
$$= \sqrt{8C_p\rho_p}\left(\frac{\beta}{2p-d+\beta}\right)^{\frac{2p-d+\beta}{2\beta}}\left(\frac{\beta}{2\beta-2p+d}\right)^{\frac{2\beta-2p+d}{2\beta}}, \quad a.s. \tag{1.43}$$

We have also obtained a variational expression for ρ. Let

$$\mathcal{E}_\beta(f,f) =: (2\pi)^{-d}\int_{\mathbb{R}^d} |\lambda|^\beta |\widehat{f}(\lambda)|^2\,\mathrm{d}\lambda, \tag{1.44}$$

and

$$\mathcal{F}_\beta = \{f \in L^2(R^d) \mid \|f\|_2 = 1,\ \mathcal{E}_\beta(f,f) < \infty\}. \tag{1.45}$$

We show in [2] that

$$\Lambda_\sigma =: \sup_{g\in\mathcal{F}_\beta}\left\{\left(\int_{(\mathbb{R}^d)^2} \frac{g^2(x)g^2(y)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{1/2} - \mathcal{E}_\beta(g,g)\right\} < \infty \tag{1.46}$$

when $0 < \sigma < \min\{2\beta, d\}$ and we derive a relation, [2, (1.20)], between ρ and Λ_σ.

We now outline the rest of the paper and explain some of the ideas we use. In Section 2 we show how to renormalize $\eta([0,t]^2_<)$ when $\beta \le \sigma < \min(3\beta/2, d)$, and establish some exponential estimates for $\eta([0,t]^2_<)$ and $\gamma([0,t]^2_<)$. The treatment adopted here is the triangular approximation developed by Varadhan [32]. In Section 3 we establish high moment asymptotics for a smoothed version of $\eta([0,t]^2_<)$, instead of using Donsker and Varadhan's large deviation approach. Here, the Fourier transform turns out to be an effective tool. Our main theorems on large deviations, Theorems 1.1 and 1.3, are proved in Section 4. Our approach is to approximate $\eta([0,t]^2_<)$ and $\gamma([0,t]^2_<)$ by their smoothed versions. The corresponding laws of the iterated logarithm are established in Section 5 (for $\eta([0,t]^2_<)$ and $\gamma([0,t]^2_<)$) and in Section 6 (for $F(t)$). Even with the LDP given in Corollary 1.5 (a short proof of which is also given in Section 6), the proof of Corollary 1.6 appears to be highly non-trivial due to the long memory possessed by the system. Our treatment involves a tail comparison and the notion of quasi-associated sequences of random variables, coming from the work of Khoshnevisan and Lewis [20]. Finally, in a short Appendix, we give details on approximation of ρ which is used in Section 3.

Conventions: We define

$$\widehat{f}(\lambda) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\lambda} f(x)\,\mathrm{d}x. \tag{1.47}$$

With this notation

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\lambda} \widehat{f}(\lambda)\, dx, \tag{1.48}$$

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda), \qquad \widehat{fg}(\lambda) = (2\pi)^{-d} \widehat{f}(\lambda) * \widehat{g}(\lambda), \tag{1.49}$$

and Parseval's identity is

$$\langle f, g\rangle = (2\pi)^{-d} \langle \widehat{f}, \widehat{g}\rangle. \tag{1.50}$$

It follows from [12, p. 156] that in our notation, for any $0 < \sigma < d$ and any $f \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi_{d-\sigma}(\lambda)\widehat{f}(\lambda)\, d\lambda = \int_{\mathbb{R}^d} \frac{1}{|x|^\sigma} f(x)\, dx. \tag{1.51}$$

2. Renormalization

We begin by proving an exponential integrability result for $\eta\left([0,1]^2_<\right)$.

Theorem 2.1. *If $0 < \sigma < \min\{\beta, d\}$, there is a $c > 0$ such that*

$$\mathbb{E} \exp\left\{c\eta\left([0,1]^2_<\right)^{\beta/\sigma}\right\} < \infty. \tag{2.1}$$

Proof. Recall that by (1.34), for each $t > 0$, $\xi(t, x) \in L^2(R^d, dx)$ almost surely. By the triangle inequality, for any $s, t > 0$,

$$\left\{\int_{\mathbb{R}^d} \xi(s+t, x)^2 dx\right\}^{1/2} \le \left\{\int_{\mathbb{R}^d} \xi(s, x)^2 dx\right\}^{1/2} + \left\{\int_{\mathbb{R}^d} [\xi(s+t, x) - \xi(s, x)]^2\, dx\right\}^{1/2}.$$

Notice that the integral

$$\begin{aligned}\int_{\mathbb{R}^d} [\xi(s+t, x) - \xi(s, x)]^2\, dx &= C \iint_{[s,s+t]^2} |X_u - X_v|^{-\sigma}\, du\, dv \\ &= C \iint_{[0,t]^2} |X_{s+u} - X_{s+v}|^{-\sigma}\, du\, dv \end{aligned} \tag{2.2}$$

is independent of $\{X_u; 0 \le u \le s\}$ and has the same distribution as

$$\int_{\mathbb{R}^d} \xi(t, x)^2 dx.$$

The process

$$\left\{\int_{\mathbb{R}^d} \xi(t, x)^2 dx\right\}^{1/2}, \quad t \ge 0, \tag{2.3}$$

is therefore sub-additive. By Theorem 1.3.5 in [8],

$$\mathbb{E} \exp\left\{\theta \left\{\int_{\mathbb{R}^d} \xi(t, x)^2 dx\right\}^{1/2}\right\} < \infty, \quad \forall \theta, t > 0,$$

and for any $\theta > 0$, the limit

$$L(\theta) \equiv \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ \theta \left\{ \int_{\mathbb{R}^d} \xi(t,x)^2 \mathrm{d}x \right\}^{1/2} \right\}$$

exists with $0 \le L(\theta) < \infty$. Taking $\theta = 1, r > L(1)$ and using Chebyshev's inequality we find that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P} \left\{ \int_{\mathbb{R}^d} \xi(t,x)^2 \mathrm{d}x \ge r^2 t^2 \right\} \le -l$$

for some $l > 0$. By the scaling property given in (1.16),

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P} \left\{ \int_{\mathbb{R}^d} \xi(1,x)^2 \mathrm{d}x \ge r^2 t^{\sigma/\beta} \right\} \le -l,$$

which leads to (2.1). □

We now show how to renormalize $\eta\left([0,t]^2_<\right)$ when $\beta \le \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$. Recall that in this case $\mathbb{E}\eta\left([0,t]^2_<\right) = \infty$. We will show how to make sense of the object formally given by (1.21).

To proceed further, let $\widetilde{X}_t$ be an independent copy of X_t and define the random measure

$$\zeta(A) = \iint_A |X_s - \widetilde{X}_t|^{-\sigma} \mathrm{d}s\mathrm{d}t \quad A \subset (\mathbb{R}^+)^2. \tag{2.4}$$

By [2, Theorem 1.1], $\zeta(A) < \infty$ a.s. for every bounded A and

$$\zeta\left([0,t]^2\right) \stackrel{d}{=} t^{2-\sigma/\beta} \zeta\left([0,1]^2\right), \quad t \ge 0. \tag{2.5}$$

Further, by [2, Theorem 1.2] there is a $\theta > 0$ such that

$$\mathbb{E} \exp\left\{ \theta \zeta\left([0,1]^2\right)^{\beta/\sigma} \right\} < \infty. \tag{2.6}$$

Note that for any $0 \le a < b < c < \infty$,

$$\eta\left([a,b]\times[b,c]\right) \stackrel{d}{=} \zeta\left([0,b-a]\times[0,c-b]\right). \tag{2.7}$$

To make sense of (1.21) we shall use Varadhan's triangular approximation (see, e.g., Proposition 6, p.194, [23]). Let $t > 0$ be fixed. Recall the subsets A_l^k defined in (1.23). By (2.5) and (2.7) we have

$$\eta(A_l^k) \stackrel{d}{=} \zeta\left(\left[0, \frac{t}{2^{k+1}}\right]^2\right) \stackrel{d}{=} 2^{-(k+1)(2-\sigma/\beta)} \zeta([0,t]^2). \tag{2.8}$$

In addition, for each $k \ge 0$, the finite sequence

$$\eta(A_l^k), \quad l = 0, 1, \ldots, 2^k - 1$$

is independent. Consequently,

$$\mathrm{Var}\left(\sum_{l=0}^{2^k-1} \eta(A_l^k) \right) = C 2^{-(3-2\sigma/\beta)k}.$$

Under $0 < \sigma < \min\{3\beta/2, d\}$, therefore,

$$\left\{\mathbb{E}\left[\sum_{k=0}^{\infty}\left|\sum_{l=0}^{2^k-1}\left(\eta(A_l^k)-\mathbb{E}\eta(A_l^k)\right)\right|\right]^2\right\}^{1/2} \le \sum_{k=0}^{\infty}\left\{\mathrm{Var}\left(\sum_{l=0}^{2^k-1}\eta(A_l^k)\right)\right\}^{1/2} < \infty.$$

Consequently, the random series

$$\sum_{k=0}^{\infty}\left\{\sum_{l=0}^{2^k-1}\left(\eta(A_l^k)-\mathbb{E}\eta(A_l^k)\right)\right\}$$

convergences in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. We may therefore define

$$\gamma\left([0,t]_<^2\right) = \sum_{k=0}^{\infty}\left\{\sum_{l=0}^{2^k-1}\left(\eta(A_l^k)-\mathbb{E}\eta(A_l^k)\right)\right\}.$$

This will be our definition of the object formally given in (1.21). As in (1.16),

$$\gamma\left([0,t]_<^2\right) \overset{d}{=} t^{2-\sigma/\beta}\gamma\left([0,1]_<^2\right), \quad t \ge 0. \tag{2.9}$$

As in Theorem 2.1, we have the following exponential integrability.

Theorem 2.2. *If* $0 < \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$, *there is a* $c > 0$ *such that*

$$\mathbb{E}\exp\left\{c\left|\gamma\left([0,1]_<^2\right)\right|^{\beta/\sigma}\right\} < \infty. \tag{2.10}$$

Proof. When $\sigma < \beta$ this follows trivially from Theorem 2.1. We can therefore assume that $p = \beta/\sigma \le 1$. Again, we use the triangular approximation based on the partition (1.23) with $t = 1$.

In view of (2.6) and (2.8), applying Lemma 1, [1] to the family of the i.i.d. sequences

$$\left\{2^{(k+1)(2-\sigma/\beta)}\left[\eta(A_l^k)-\mathbb{E}\eta(A_l^k)\right]; l = 0, 1, \dots, 2^k-1\right\}, \quad k = 0, 1, \dots$$

gives that for some $\theta > 0$

$$\sup_{k\ge 0}\mathbb{E}\exp\left\{\theta\left|2^{-k/2}\sum_{l=0}^{2^k-1}2^{k(2-\sigma/\beta)}\left[\eta(A_l^k)-\mathbb{E}\eta(A_l^k)\right]\right|^{\beta/\sigma}\right\} < \infty,$$

or

$$e^C \equiv \sup_{k\ge 0}\mathbb{E}\exp\left\{2^{ak}\theta\left|\sum_{l=0}^{2^k-1}\left[\eta(A_l^k)-\mathbb{E}\eta(A_l^k)\right]\right|^{\beta/\sigma}\right\} < \infty \tag{2.11}$$

where

$$a = \frac{3\beta}{2\sigma} - 1 > 0.$$

For each $N \geq 1$ set

$$b_1 = \theta, \qquad b_N = \theta \prod_{j=2}^{N} \left(1 - 2^{-a(j-1)}\right), \quad N = 2, 3, \ldots.$$

By Hölder's inequality and the triangular inequality

$$\begin{aligned}
&\mathbb{E} \exp \left\{ b_N \left| \sum_{k=0}^{N} \sum_{l=0}^{2^k-1} \left[\eta(A_l^k) - \mathbb{E}\eta(A_l^k) \right] \right|^{\beta/\sigma} \right\} \\
&\leq \left[\mathbb{E} \exp \left\{ b_{N-1} \left| \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} \left[\eta(A_l^k) - \mathbb{E}\beta(A_l^k) \right] \right|^{\eta/\sigma} \right\} \right]^{1-2^{-a(N-1)}} \\
&\quad \times \left[\mathbb{E} \exp \left\{ 2^{a(N-1)} b_N \left| \sum_{l=0}^{2^N-1} \left[\eta(A_l^N) - \mathbb{E}\eta(A_l^N) \right] \right|^{\beta/\sigma} \right\} \right]^{2^{-a(N-1)}}.
\end{aligned}$$

Notice that $b_N \leq \theta$. By (2.11) we have

$$\begin{aligned}
&\mathbb{E} \exp \left\{ b_N \left| \sum_{k=0}^{N} \sum_{l=0}^{2^k-1} \left[\eta(A_l^k) - \mathbb{E}\eta(A_l^k) \right] \right|^{\beta/\sigma} \right\} \\
&\leq \exp\left\{ C 2^{-a(N-1)} \right\} \mathbb{E} \exp \left\{ b_{N-1} \left| \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} \left[\eta(A_l^k) - \mathbb{E}\eta(A_l^k) \right] \right|^{\beta/\sigma} \right\}.
\end{aligned}$$

Repeating the above procedure gives

$$\begin{aligned}
&\mathbb{E} \exp \left\{ b_N \left| \sum_{k=0}^{N} \sum_{l=0}^{2^k-1} \left[\eta(A_l^k) - \mathbb{E}\eta(A_l^k) \right] \right|^{\beta/\sigma} \right\} \\
&\leq \exp\left\{ C \sum_{k=0}^{N} 2^{-a(k-1)} \right\} \leq \exp\left\{ C \left(1 - 2^{-a}\right)^{-1} \right\} < \infty.
\end{aligned}$$

Observe that

$$b_\infty = \theta \prod_{j=2}^{\infty} \left(1 - 2^{-a(j-1)}\right) > 0.$$

By Fatou's lemma, letting $N \to \infty$ we have

$$\mathbb{E} \exp \left\{ b_\infty \left| \gamma\left([0,1]_<^2\right) \right|^{\beta/\sigma} \right\} \leq \exp\left\{ C' \left(1 - 2^{-a}\right)^{-1} \right\} < \infty. \quad \square$$

Among other things, we now show that the family

$$\left\{ \gamma\left([0,t]_<^2\right) t \geq 0 \right\}$$

has a continuous version.

Lemma 2.3. *Assume* $0 < \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$. *For any* $T > 0$ *there is a* $c = c(T) > 0$ *such that*

$$\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \mathbb{E}\exp\left\{c\frac{\left|\gamma\left([0,t]_<^2\right)-\gamma\left([0,s]_<^2\right)\right|^{\beta/\sigma}}{|t-s|^{(2\beta-\sigma)/(2\sigma)}}\right\} < \infty.$$

Proof. For $0 \le s < t \le T$,

$$\gamma\left([0,t]_<^2\right)-\gamma\left([0,s]_<^2\right)=\gamma\left([s,t]_<^2\right)+\gamma\left([0,s]\times[s,t]\right).$$

Notice that

$$\gamma\left([s,t]_<^2\right)\overset{d}{=}(t-s)^{2-\sigma/\beta}\gamma([0,1]_<^2).$$

By Theorem 2.2, there is a $c_1 > 0$ such that

$$\sup_{\substack{s,t\in[0,T]\\ s<t}} \mathbb{E}\exp\left\{c_1\frac{\left|\gamma\left([s,t]_<^2\right)\right|^{\beta/\sigma}}{|t-s|^{(2\beta-\sigma)/\sigma}}\right\} < \infty.$$

In addition, by (2.7)

$$\gamma\left([0,s]\times[s,t]\right)\overset{d}{=}\zeta\left([0,s]\times[0,t-s]\right)-\mathbb{E}\zeta\left([0,s]\times[0,t-s]\right).$$

In view of (1.31),

$$\zeta\left([0,s]\times[0,t-s]\right)=C^{-1}\int_{\mathbb{R}^d}\xi(s,x)\tilde{\xi}(t-s,x)\mathrm{d}x,$$

where

$$\xi(t,x)=\int_0^t|X_u-x|^{-\frac{\sigma+d}{2}}\mathrm{d}u \quad\text{and}\quad \tilde{\xi}(t,x)=\int_0^t|\widetilde{X}_u-x|^{-\frac{\sigma+d}{2}}\mathrm{d}u.$$

By independence, for any integer $m \ge 1$,

$$\begin{aligned}
&\mathbb{E}\left[\zeta\left([0,s]\times[0,t-s]\right)^m\right]\\
&\quad= C^{-m}\int_{(\mathbb{R}^d)^m}\mathrm{d}x_1\cdots\mathrm{d}x_m\left[\mathbb{E}\prod_{k=1}^m\xi(s,x_k)\right]\left[\mathbb{E}\prod_{k=1}^m\xi(t-s,x_k)\right]\\
&\quad\le C^{-m}\left\{\int_{(\mathbb{R}^d)^m}\mathrm{d}x_1\cdots\mathrm{d}x_m\left[\mathbb{E}\prod_{k=1}^m\xi(s,x_k)\right]^2\right\}^{1/2}\\
&\qquad\times\left\{\int_{(\mathbb{R}^d)^m}\mathrm{d}x_1\cdots\mathrm{d}x_m\left[\mathbb{E}\prod_{k=1}^m\xi(t-s,x_k)\right]^2\right\}^{1/2}\\
&\quad=\left\{\mathbb{E}\left[\zeta\left([0,s]^2\right)^m\right]\right\}^{1/2}\left\{\mathbb{E}\left[\zeta\left([0,t-s]^2\right)^m\right]\right\}^{1/2}\\
&\quad= s^{\frac{2\beta-\sigma}{2\beta}m}(t-s)^{\frac{2\beta-\sigma}{2\beta}m}\left(\mathbb{E}\left[\zeta\left\{[0,1]^2\right)^m\right]\right\}.
\end{aligned}$$

Checking (2.6), there is $c_2 > 0$ such that

$$\sup_{\substack{s,t\in[0,T]\\ s<t}} \mathbb{E}\exp\left\{c_2 \frac{|\gamma([0,s]\times[s,t])|^{\beta/\sigma}}{|t-s|^{\frac{2\beta-\sigma}{2\sigma}}}\right\} < \infty.$$

The proof is now complete. □

Recall that a function Ψ: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a Young function if it is convex, increasing and satisfies $\Psi(0) = 0$, $\lim_{x\to\infty} \Psi(x) = \infty$. The Orlicz space $\mathcal{L}_\Psi(\Omega, \mathcal{A}, \mathbb{P})$ is defined as the linear space of all random variables X on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$\|X\|_\Psi = \inf\{c > 0; \mathbb{E}\Psi(c^{-1}|X|) \le 1\} < \infty.$$

It is known that $\|\cdot\|_\Psi$ defines a norm (called Orlicz norm) under which $\mathcal{L}_\Psi(\Omega, \mathcal{A}, \mathbb{P})$ becomes a Banach space.

We now choose the Young function $\Psi(\cdot)$ such that $\Psi(x) \sim \exp\{x^{\beta/\sigma}\}$ as $x \to \infty$. By Lemma 2.3, for any $T > 0$ there is $c = c(d, \beta, \sigma, T) > 0$ such that

$$\left\|\gamma\left([0,t]^2_<\right) - \gamma\left([0,s]^2_<\right)\right\|_\Psi \le c|t-s|^{(2\beta-\sigma)/(2\beta)}, \quad s,t \in [0,T]. \tag{2.12}$$

By Lemma 9 in [10], the family $\left\{\gamma\left([0,t]^2_<\right) t \ge 0\right\}$ has a continuous version and in the future we will always use this version. Furthermore, for any $0 < b < \frac{2\beta-\sigma}{2\beta}$,

$$\left\|\sup_{0\le s<t\le 1} \frac{\gamma\left([0,t]^2_<\right) - \gamma\left([0,s]^2_<\right)}{|t-s|^b}\right\|_\Psi < \infty.$$

Equivalently, there is $c > 0$ such that

$$\mathbb{E}\exp\left\{c \sup_{0\le s<t\le 1} \frac{\left|\gamma\left([0,t]^2_<\right) - \gamma\left([0,s]^2_<\right)\right|^{\beta/\sigma}}{|t-s|^{b\beta/\sigma}}\right\} < \infty. \tag{2.13}$$

A natural question is: what happens if $3\beta/2 \le \sigma < d$? Recall [2] that given an independent copy $\widetilde{X}_t$ of X_t, the integral

$$\int_0^t \int_0^t \frac{\mathrm{d}r\mathrm{d}s}{|X_r - \widetilde{X}_s|^\sigma}$$

is finite if and only if $\sigma < \min\{2\beta, d\}$. By (2.7), $\gamma([a,b]\times[b,c]) = \infty$ for any $0 \le a < b < c < \infty$ if $\sigma \ge 2\beta$. So the question to ask is: what happens when $3\beta/2 \le \sigma < \min\{2\beta, d\}$. In light of the existing results, [34,5], for self-intersection local times, we believe that $\gamma([0,t]^2_<)$ is not definable when $3\beta/2 \le \sigma < \min\{2\beta, d\}$, in the sense that for any reasonably good probability density $h(x)$ on $\mathbb{R}^d$,

$$\lim_{\epsilon\to 0^+}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{L(t,x,\epsilon)L(t,y,\epsilon)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y - \mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{L(t,x,\epsilon)L(t,y,\epsilon)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right\} = \infty$$

where

$$L(t,x,\epsilon) = \epsilon^{-d}\int_0^t h\left(\epsilon^{-1}(X_s - x)\right)\mathrm{d}s \quad x \in \mathbb{R}^d\ t \ge 0.$$

We also believe that in this case there is a deterministic function $\varphi(\epsilon)$ depending on (d, β, σ), with $\varphi(\epsilon) \to 0^+$ as $\epsilon \to 0^+$, such that

$$\varphi(\epsilon)\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{L(t,x,\epsilon)L(t,y,\epsilon)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y-\mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{L(t,x,\epsilon)L(t,y,\epsilon)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right\}$$

converges in law to a normal distribution. We leave this problem for future study.

3. High moment asymptotics for the smoothed version

Under $0 < \sigma < \min\{\beta, d\}$, by Fourier inversion and (1.51)

$$\eta\left([0,t]^2\right)=\int_{\mathbb{R}^d}\varphi_{d-\sigma}(\lambda)\left|\int_0^t e^{i\lambda\cdot X_s}\mathrm{d}s\right|^2\mathrm{d}\lambda. \tag{3.1}$$

It would be possible to obtain a lower bound for $\eta\left([0,t]^2\right)$ similar to (3.14) below, by slightly modifying our argument. There are several reasons that we deal with the smoothed version $\eta_{\alpha,\epsilon}\left([0,t]^2\right)$ (defined in (3.8) below), rather than with $\eta\left([0,t]^2\right)$. First, $\eta\left([0,t]^2\right)$ is not defined for $\beta \le \sigma < \min\{3\beta/2, d\}$, while $\eta_{\alpha,\epsilon}\left([0,t]^2\right)$ is. The high moment asymptotics for $\eta_{\alpha,\epsilon}\left([0,t]^2\right)$ will be used for approximating $\gamma([0,t]^2)$ in the proof of Theorem 1.3. Secondly, even when $0 < \sigma < \min\{\beta, d\}$, our approach does not allow us to deal directly with the λ-integral on the right-hand side of (3.1) in establishing the upper bound. A discretization procedure is needed, replacing the λ-integral by a sum over lattice points. Since $\varphi_{d-\sigma}(0) = \infty$, we need to replace $\varphi_{d-\sigma}(\lambda)$ by $C_{d,\sigma}(\alpha + |\lambda|^2)^{-(d-\sigma)/2}$ with $\alpha > 0$ being small. For localizing $\varphi_{d-\sigma}$, we introduce the probability density function

$$h(x)=C_0^{-1}\prod_{j=1}^d\left(\frac{2\sin x_j}{x_j}\right)^2,\quad x=(x_1,\ldots,x_d)\in\mathbb{R}^d \tag{3.2}$$

where $C_0 > 0$ is the normalizing constant:

$$C_0=\int_{\mathbb{R}^d}\prod_{j=1}^d\left(\frac{2\sin x_k}{x_k}\right)^2\mathrm{d}x_1\cdots\mathrm{d}x_d.$$

Clearly, h is symmetric. One can verify that its Fourier transform $\widehat{h}$ is

$$\widehat{h}(\lambda)=\int_{\mathbb{R}^d}h(x)e^{i\lambda\cdot x}\mathrm{d}x=C_0^{-1}(2\pi)^d\left(1_{[-1,1]^d}*1_{[-1,1]^d}\right)(\lambda).$$

In particular, $\widehat{h}$ is non-negative, continuous and most importantly, $\widehat{h}$ has the compact support in the set $[-2,2]^d$. As characteristic function,

$$\widehat{h}(\lambda)\le\widehat{h}(0)=1. \tag{3.3}$$

Our final replacement of $\varphi_{d-\sigma}(\lambda)$ in this section is

$$\wp_{\alpha,\epsilon}(\lambda)=\frac{C_{d,\sigma}\widehat{h}^2(\epsilon\lambda)}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}}. \tag{3.4}$$

Note that by (3.3)

$$\wp_{\alpha,\epsilon}(\lambda)\le\wp_{0,\epsilon}(\lambda)\le\varphi_{d-\sigma}(\lambda) \tag{3.5}$$

with $\alpha, \epsilon > 0$, and that the localizer $\widehat{h}(\epsilon\lambda)$ is the Fourier transform (or characteristic function) of the probability density function

$$h_\epsilon(x) = \epsilon^{-d} h(\epsilon^{-1}x), \quad x \in \mathbb{R}^d \tag{3.6}$$

which has a high concentration near 0 for small $\epsilon > 0$.

Set

$$\theta_{\alpha,\epsilon}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\lambda} \wp_{\alpha,\epsilon}(\lambda)\,\mathrm{d}\lambda. \tag{3.7}$$

Our attention in this section is mainly on the smoothed random measure

$$\eta_{\alpha,\epsilon}(A) = \int\int_A \theta_{\alpha,\epsilon}(X_{s_1} - X_{s_2})\,\mathrm{d}s_1\,\mathrm{d}s_2 \quad A \subset R_+^2. \tag{3.8}$$

Throughout this section we assume that

$$0 < \sigma < \min\{2\beta, d\}. \tag{3.9}$$

Let τ be an independent mean-1 exponential,

$$\begin{aligned}\eta_{\alpha,\epsilon}\left([0,\tau]^2\right) &= \int_0^\tau\int_0^\tau\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(X_{s_1}-X_{s_2})\cdot\lambda} \wp_{\alpha,\epsilon}(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= \int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda)\left|\int_0^\tau \mathrm{e}^{\mathrm{i}\lambda\cdot X_s}\mathrm{d}s\right|^2 \mathrm{d}\lambda.\end{aligned} \tag{3.10}$$

Let $f(\lambda)$ be a continuous and strictly positive function on $\mathbb{R}^d$ such that

$$f(-\lambda) = f(\lambda) \quad \text{and} \quad \int_{\mathbb{R}^d} f^2(\lambda)\wp_{\alpha,\epsilon}(\lambda)\mathrm{d}\lambda = 1.$$

By the Cauchy–Schwartz inequality

$$\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{1/2} \ge \left|\int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) f(\lambda)\left[\int_0^\tau \mathrm{e}^{\mathrm{i}\lambda\cdot X_s}\mathrm{d}s\right]\mathrm{d}\lambda\right|.$$

Hence,

$$\begin{aligned}&\mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2}\right] \\ &\quad\ge \int_{\mathbb{R}^n} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n\left(\prod_{k=1}^n \wp_{\alpha,\epsilon}(\lambda_k) f(\lambda_k)\right)\mathbb{E}\int_{[0,\tau]^n} \mathrm{d}s_1\cdots\mathrm{d}s_n \exp\left\{\mathrm{i}\sum_{k=1}^n \lambda_k\cdot X_{s_k}\right\}.\end{aligned}$$

Let Σ_n be the permutation group on the set $\{1,\ldots,n\}$ and adopt the notation

$$[0,t]_<^n = \left\{(s_1,\ldots,s_n) \in [0,t]^n;\, s_1 < \cdots < s_n\right\}.$$

We have

$$\begin{aligned}&\mathbb{E}\int_{[0,\tau]^n} \mathrm{d}s_1\cdots\mathrm{d}s_n \exp\left\{\mathrm{i}\sum_{k=1}^n \lambda_k\cdot X_{s_k}\right\} \\ &\quad= \int_0^\infty \mathrm{d}t\mathrm{e}^{-t}\int_{[0,t]^n} \mathrm{d}s_1\cdots\mathrm{d}s_n\mathbb{E}\exp\left\{\mathrm{i}\sum_{k=1}^n \lambda_k\cdot X_{s_k}\right\}\end{aligned}$$

$$= \sum_{\sigma\in\Sigma_n} \int_0^\infty \mathrm{d}t \mathrm{e}^{-t} \int_{[0,t]^n_<} \mathrm{d}s_1\cdots\mathrm{d}s_n \mathbb{E}\exp\left\{\mathrm{i}\sum_{k=1}^n \lambda_{\sigma(k)}\cdot X_{s_k}\right\}.$$

Write

$$\sum_{k=1}^n \lambda_{\sigma(k)}\cdot X_{s_k} = \sum_{k=1}^n \left(\sum_{j=k}^n \lambda_{\sigma(j)}\right)\left(X_{s_k} - X_{s_{k-1}}\right).$$

By independence,

$$\mathbb{E}\exp\left\{\mathrm{i}\sum_{k=1}^n \lambda_{\sigma(k)}\cdot X_{s_k}\right\} = \exp\left\{-\sum_{k=1}^n (s_k - s_{k-1})\psi\left(\sum_{j=k}^n \lambda_{\sigma(j)}\right)\right\}.$$

Therefore,

$$\mathbb{E}\int_{[0,\tau]^n} \mathrm{d}s_1\cdots\mathrm{d}s_n \exp\left\{\mathrm{i}\sum_{k=1}^n \lambda_k\cdot X_{s_k}\right\} = \sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\left(\sum_{j=k}^n \lambda_{\sigma(j)}\right)$$

where $Q(\lambda) = (1+\psi(\lambda))^{-1}$.

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2}\right] \\
&\quad\geq \sum_{\sigma\in\Sigma_n}\int_{\mathbb{R}^n} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n \left(\prod_{k=1}^n \wp_{\alpha,\epsilon}(\lambda_k) f(\lambda_k)\right)\prod_{k=1}^n Q\left(\sum_{j=k}^n \lambda_{\sigma(j)}\right) \\
&\quad= n!\int_{\mathbb{R}^n} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n \left(\prod_{k=1}^n \wp_{\alpha,\epsilon}(\lambda_k) f(\lambda_k)\right)\prod_{k=1}^n Q\left(\sum_{j=k}^n \lambda_j\right).
\end{aligned} \tag{3.11}$$

By a change of variables

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n \left(\prod_{k=1}^n \wp_{\alpha,\epsilon}(\lambda_k) f(\lambda_k)\right)\prod_{k=1}^n Q\left(\sum_{j=k}^n \lambda_j\right) \\
&\quad= \int_{\mathbb{R}^n} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n \left(\prod_{k=1}^n \wp_{\alpha,\epsilon}(\lambda_k-\lambda_{k-1}) f(\lambda_k-\lambda_{k-1}) Q(\lambda_k)\right),
\end{aligned}$$

where we follow the convention $\lambda_0 = 0$.

Applying an argument based on the spectral representation of self-adjoint operators in L^2 (see (3.7)–(3.10) in [2]) to the right-hand side,

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{\mathbb{R}^n} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n \left(\prod_{k=1}^n \wp_{\alpha,\epsilon}(\lambda_k-\lambda_{k-1}) f(\lambda_k-\lambda_{k-1}) Q(\lambda_k)\right) \\
&\quad\geq \log\sup_{\|g\|_2=1}\int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) f(\lambda)\left[\int_{\mathbb{R}^d}\sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)\mathrm{d}\gamma\right]\mathrm{d}\lambda.
\end{aligned} \tag{3.12}$$

Write

$$\rho_{\alpha,\epsilon} = \sup_{||g||_2=1} \int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) \mathrm{d}\gamma \right]^2 \mathrm{d}\lambda. \tag{3.13}$$

Taking the supremum over f on the right-hand side of (3.12) leads to the conclusion that

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2} \right] \geq \frac{1}{2} \log \rho_{\alpha,\epsilon}. \quad \square \tag{3.14}$$

We note for future reference that

$$\lim_{\alpha,\epsilon\to 0} \rho_{\alpha,\epsilon} = \rho. \tag{3.15}$$

To see this, we first write

$$\begin{aligned} 0 \leq \rho - \rho_{\alpha,\epsilon} \leq & \sup_{||g||_2=1} \int_{\mathbb{R}^d} \left(1 - \frac{\widehat{h}^2(\epsilon\lambda)|\lambda|^{d-\sigma}}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}} \right) \varphi_{d-\sigma}(\lambda) \\ & \times \left[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) \mathrm{d}\gamma \right]^2 \mathrm{d}\lambda. \end{aligned} \tag{3.16}$$

For $M > 0$, let

$$B(\alpha,\epsilon,M) = \sup_{M^{-1}\leq|\lambda|\leq M} \left| 1 - \frac{\widehat{h}^2(\epsilon\lambda)|\lambda|^{d-\sigma}}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}} \right|. \tag{3.17}$$

Then

$$\begin{aligned} & \left| 1 - \frac{\widehat{h}^2(\epsilon\lambda)|\lambda|^{d-\sigma}}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}} \right| \varphi_{d-\sigma}(\lambda) \\ & \quad \leq B(\alpha,\epsilon,M)\varphi_{d-\sigma}(\lambda) + M^{-\delta}(\varphi_{d-\sigma-\delta}(\lambda) + \varphi_{d-\sigma+\delta}(\lambda)). \end{aligned} \tag{3.18}$$

Using this and (3.16), it follows from [2, Lemma 1.6] that for $\delta > 0$ sufficiently small

$$0 \leq \rho - \rho_{\alpha,\epsilon} \leq CB(\alpha,\epsilon,M) + CM^{-\delta}. \tag{3.19}$$

Note that by (3.3) we have $\lim_{\alpha,\epsilon\to 0} B(\alpha,\epsilon,M) = 0$ for any $M > 0$. The limit (3.15) then follows by first taking $\alpha,\epsilon \to 0$ and then $M \to \infty$.

Note also for future reference that for any $c > 0$ and $t > 0$,

$$\begin{aligned} \eta_{\alpha,\epsilon}\left([0,ct]^2_<\right) &= \int_0^{ct}\int_0^{s_2}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(X_{s_1}-X_{s_2})\cdot\lambda} \frac{C_{d,\sigma}\widehat{h}^2(\epsilon\lambda)}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}} \,\mathrm{d}\lambda\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= c^2 \int_0^{t}\int_0^{s_2}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(X_{cs_1}-X_{cs_2})\cdot\lambda} \frac{C_{d,\sigma}\widehat{h}^2(\epsilon\lambda)}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}} \,\mathrm{d}\lambda\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &\overset{d}{=} c^2 \int_0^{t}\int_0^{s_2}\int_{\mathbb{R}^d} \mathrm{e}^{i(X_{s_1}-X_{s_2})\cdot c^{1/\beta}\lambda} \frac{C_{d,\sigma}\widehat{h}^2(\epsilon\lambda)}{(\alpha+|\lambda|^2)^{(d-\sigma)/2}} \,\mathrm{d}\lambda\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= c^{2-d/\beta} \int_0^{t}\int_0^{s_2}\int_{\mathbb{R}^d} \mathrm{e}^{i(X_{s_1}-X_{s_2})\cdot\lambda} \frac{C_{d,\sigma}\widehat{h}^2(\epsilon\lambda/c^{1/\beta})}{(\alpha+|\lambda/c^{1/\beta}|^2)^{(d-\sigma)/2}} \,\mathrm{d}\lambda\,\mathrm{d}s_1\,\mathrm{d}s_2 \end{aligned}$$

$$= c^{2-\sigma/\beta} \int_0^t \int_0^{s_2} \int_{\mathbb{R}^d} e^{i(X_{s_1}-X_{s_2})\cdot\lambda} \frac{C_{d,\sigma} \widehat{h}^2(\epsilon\lambda/c^{1/\beta})}{(\alpha c^{2/\beta} + |\lambda|^2)^{(d-\sigma)/2}} \, d\lambda \, ds_1 \, ds_2$$

$$= c^{2-\sigma/\beta} \eta_{\alpha c^{2/\beta},\, \epsilon c^{-1/\beta}} \left([0, t]^2_< \right). \tag{3.20}$$

Lemma 3.1.

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \mathbb{E}\left[\eta_{\alpha,\epsilon} \left([0, \tau]^2 \right)^{n/2} \right] \le \frac{1}{2} \log \rho_{\alpha,\epsilon}. \tag{3.21}$$

Proof. Let

$$\bar{\theta}_{\alpha,\epsilon}(x) = \int_{\mathbb{R}^d} e^{ix\cdot\lambda} \frac{C_{d,\sigma} \widehat{h}(\epsilon\lambda)}{(\alpha + |\lambda|^2)^{(d-\sigma)/2}} \, d\lambda, \tag{3.22}$$

and

$$\psi^x_{\alpha,\epsilon}(A) = \int\int_A \bar{\theta}_{\alpha,\epsilon}(X_{s_1} - X_{s_2} - x) \, ds_1 \, ds_2. \tag{3.23}$$

Then

$$\psi^x_{\alpha,\epsilon}\left([0, \tau]^2 \right) = \int_0^\tau \int_0^\tau \int_{\mathbb{R}^d} e^{i(X_{s_1}-X_{s_2}-x)\cdot\lambda} \frac{C_{d,\sigma} \widehat{h}(\epsilon\lambda)}{(\alpha + |\lambda|^2)^{(d-\sigma)/2}} \, d\lambda \, ds_1 \, ds_2$$

$$= \int_{\mathbb{R}^d} e^{-ix\cdot\lambda} \frac{C_{d,\sigma} \widehat{h}(\epsilon\lambda)}{(\alpha + |\lambda|^2)^{(d-\sigma)/2}} \left| \int_0^\tau e^{i\lambda\cdot X_s} ds \right|^2 d\lambda, \tag{3.24}$$

so that

$$\eta_{\alpha,\epsilon}\left([0, \tau]^2 \right) = \int h_\epsilon(x) \psi^x_{\alpha,\epsilon}\left([0, \tau]^2 \right) dx. \tag{3.25}$$

Let $M > 0$ be a large but fixed number. Define the random measure $\tilde{\psi}^x_{\alpha,\epsilon}(\cdot)$ as

$$\tilde{\psi}^x_{\alpha,\epsilon}(A) = \sum_{y\in\mathbb{Z}^d} \psi^{yM+x}_{\alpha,\epsilon}(A)$$

and write

$$\tilde{h}_\epsilon(x) = \sum_{y\in\mathbb{Z}^d} h_\epsilon(yM + x).$$

Then

$$\eta_{\alpha,\epsilon}([0, \tau]^2) = \sum_{y\in\mathbb{Z}^d} \int_{[0,M]^2} h_\epsilon(yM + x) \psi^{yM+x}_{\alpha,\epsilon}\left([0, \tau]^2 \right)$$

$$\le \int_{[0,M]^d} \tilde{h}_\epsilon(z) \tilde{\psi}^z_{\alpha,\epsilon}([0, \tau]^2) dz.$$

Here we used the fact that $\bar{\theta}_{\alpha,\epsilon}(x)$, being the convolution of the positive function h_ϵ and a Bessel kernel, is itself positive, and hence so is $\psi^x_{\alpha,\epsilon}\left([0, \tau]^2 \right)$. Recall, [12, p. 294] or [30, p. 131, (v)] that

$$\frac{\int_{R^d} e^{i\,x\cdot\lambda}}{(1 + |\lambda|^2)^{(d-\sigma)/2}} \, d\lambda = \frac{\int_0^\infty e^{-|x|^2/4t} e^{-t} t^{-1-\sigma/2} \, dt}{\pi^{d/2} \Gamma(d - \sigma)}.$$

In addition, by Parseval's identity,

$$\int_{[0,M]^d} \tilde{h}_\epsilon(z)\tilde{\psi}^z_{\alpha,\epsilon}([0,\tau]^2)\mathrm{d}z = M^{-d}\sum_{y\in\mathbb{Z}^d}\left(\int_{[0,M]^d}\tilde{h}_\epsilon(x)\exp\left\{-\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x\right)$$
$$\times\left(\int_{[0,M]^d}\tilde{\psi}^x_{\alpha,\epsilon}([0,\tau]^2)\exp\left\{\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x\right).$$

By the periodicity of $\tilde{h}_\epsilon$,

$$\begin{aligned}
&\int_{[0,M]^d}\tilde{h}_\epsilon(x)\exp\left\{-\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x\\
&\quad=\sum_{z\in\mathbb{Z}^d}\int_{[0,M]^d}h_\epsilon(zM+x)\exp\left\{-\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x\\
&\quad=\sum_{z\in\mathbb{Z}^d}\int_{[0,M]^d}h_\epsilon(zM+x)\exp\left\{-\mathrm{i}\frac{2\pi}{M}\,(y\cdot(zM+x))\right\}\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^d}h_\epsilon(x)\exp\left\{-\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x=\hat{h}\left(\frac{2\pi\epsilon}{M}y\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{[0,M]^d}\tilde{\psi}^x_{\alpha,\epsilon}([0,\tau]^2)\exp\left\{\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^d}\psi^x_{\alpha,\epsilon}([0,\tau]^2)\exp\left\{-\mathrm{i}\frac{2\pi}{M}(y\cdot x)\right\}\mathrm{d}x\\
&\quad=(2\pi)^d\frac{C_{d,\sigma}\widehat{h}\left(\frac{2\pi\epsilon}{M}y\right)}{(\alpha+|\frac{2\pi}{M}y|^2)^{(d-\sigma)/2}}\left|\int_0^\tau\exp\left\{\mathrm{i}\frac{2\pi}{M}\,(y\cdot X_s)\right\}\mathrm{d}s\right|^2,
\end{aligned}$$

where the last step follows by (3.24).

Summarizing the computation,

$$\begin{aligned}
\eta_{\alpha,\epsilon}([0,\tau]^2)&\le\left(\frac{2\pi}{M}\right)^d\sum_{y\in\mathbb{Z}^d}\wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}y\right)\left|\int_0^\tau\exp\left\{\mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\}\mathrm{d}s\right|^2\\
&=\left(\frac{2\pi}{M}\right)^d\sum_{y\in E}\wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}y\right)\left|\int_0^\tau\exp\left\{\mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\}\mathrm{d}s\right|^2,
\end{aligned}\tag{3.26}$$

where

$$E=\mathbb{Z}^d\cap\left[-\frac{M}{\pi\epsilon},\frac{M}{\pi\epsilon}\right],$$

and the last step follows from the fact that $\hat{h}(\lambda)$ is supported on $[-2,2]^d$.

Write

$$\pi_{\alpha,\epsilon,M}(y)=\wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}y\right).$$

Let $\mathcal{H}$ be the finite dimensional U-space of the complex-valued functions $g(y)$ on E with

$$\|g\| = \left\{ \sum_{y\in E} |g(y)|^2 \pi_{\alpha,\epsilon,M}(y) \right\}^{1/2}.$$

Let the subset $\mathcal{U} \subset \mathcal{H}$ be defined by the property

$$\overline{g(y)} = g(-y) \quad y \in E.$$

Let $\delta > 0$ be a small but fixed number. There are $f_1, \ldots, f_l \in \mathcal{U}$ with norm 1 such that

$$\|g\| \le (1+\delta) \max_{1\le j\le l} |\langle f_j, g\rangle| \quad g \in \mathcal{U}.$$

In particular,

$$\begin{aligned}
&\left\{ \sum_{y\in E} \pi_{\alpha,\epsilon,M}(y) \left| \int_0^\tau \exp\left\{ \mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\} \mathrm{d}s \right|^2 \right\}^{1/2} \\
&\quad \le (1+\delta) \max_{1\le j\le l} \left| \sum_{y\in E} \pi_{\alpha,\epsilon,M}(y) f_j(y) \int_0^\tau \exp\left\{ \mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\} \mathrm{d}s \right| \\
&\quad = (1+\delta) \max_{1\le j\le l} \left| \sum_{y\in \mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(y) f_j(y) \int_0^\tau \exp\left\{ \mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\} \mathrm{d}s \right|.
\end{aligned}$$

By (3.26),

$$\begin{aligned}
&\mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2} \right] \\
&\quad \le (1+\delta)^n \left(\frac{2\pi}{M}\right)^{nd/2} \sum_{j=1}^{l} \mathbb{E} \left| \sum_{y\in \mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(y) f_j(y) \int_0^\tau \exp\left\{ \mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\} \mathrm{d}s \right|^n. \quad (3.27)
\end{aligned}$$

We now intend to establish

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \mathbb{E} \left| \sum_{y\in \mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(y) f_j(y) \int_0^\tau \exp\left\{ \mathrm{i}\frac{2\pi}{M}(y\cdot X_s)\right\} \mathrm{d}s \right|^n \\
&\quad \le \frac{1}{2} \log \rho_{\alpha,\epsilon,M} \qquad (3.28)
\end{aligned}$$

for each $j = 1, \ldots, l$, where

$$\begin{aligned}
\rho_{\alpha,\epsilon,M} &= \sup_{|g|_2=1} \sum_{x\in\mathbb{Z}^d} \wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}x\right) \\
&\quad \times \left[\sum_{y\in\mathbb{Z}^d} \sqrt{Q\left(\frac{2\pi}{M}(x+y)\right)} \sqrt{Q\left(\frac{2\pi}{M}y\right)} g(x+y) g(y) \right]^2, \qquad (3.29)
\end{aligned}$$

where the supremum is over all functions $g(x)$ on $\mathbb{Z}^d$ satisfying

$$|g|_2 \equiv \left\{ \sum_{y \in \mathbb{Z}^d} g^2(y) \right\}^{1/2} = 1.$$

Write $f = f_j$. Using estimates of the form

$$\mathbb{E}|X|^{2m+1} \le \left\{ \mathbb{E}|X|^{2m} \right\}^{1/2} \left\{ \mathbb{E}|X|^{2(m+1)} \right\}^{1/2},$$

we need only consider the case $n = 2m$. That is, we need only to show that

$$\begin{aligned} &\limsup_{m\to\infty} \frac{1}{2m} \log \frac{1}{(2m)!} \mathbb{E} \left| \sum_{y \in \mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(y) f_j(y) \int_0^\tau \exp\left\{ \mathrm{i} \frac{2\pi}{M} (y \cdot X_s) \right\} \mathrm{d}s \right|^{2m} \\ &\quad \le \frac{1}{2} \log \rho_{\alpha,\epsilon,M} \end{aligned} \tag{3.30}$$

where the last line follows from the fact that the f_j are supported on E.

Indeed,

$$\begin{aligned} &\mathbb{E} \left| \sum_{y \in \mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(x) f(y) \int_0^\tau \exp\left\{ \mathrm{i} \frac{2\pi}{M} (y \cdot X_s) \right\} \mathrm{d}s \right|^{2m} \\ &\quad = \sum_{y_1,\ldots,y_{2m} \in \mathbb{Z}^d} \left(\prod_{k=1}^m \pi_{\alpha,\epsilon,M}(y_k) \pi_{\alpha,\epsilon,M}(y_{m+k}) f(y_k) \overline{f(y_{m+k})} \right) \\ &\qquad \times \mathbb{E} \int_{[0,\tau]^{2m}} \mathrm{d}s_1 \cdots \mathrm{d}s_{2m} \exp\left\{ \mathrm{i} \frac{2\pi}{M} \sum_{k=1}^m \left[(y_k \cdot X_{s_k}) - (y_{m+k} \cdot X_{s_{m+k}}) \right] \right\} \\ &\quad = \sum_{y_1,\ldots,y_{2m} \in \mathbb{Z}^d} \left(\prod_{k=1}^{2m} \pi_{\alpha,\epsilon,M}(y_k) f(y_k) \right) \\ &\qquad \times \mathbb{E} \int_{[0,\tau]^{2m}} \mathrm{d}s_1 \cdots \mathrm{d}s_{2m} \exp\left\{ \mathrm{i} \frac{2\pi}{M} \sum_{k=1}^{2m} (y_k \cdot X_{s_k}) \right\}. \end{aligned}$$

Similar to the computation for (3.12), (with $n = 2m$), the right-hand side is equal to

$$(2m)! \sum_{y_1,\ldots,y_{2m} \in \mathbb{Z}^d} \left(\prod_{k=1}^{2m} \pi_{\alpha,\epsilon,M}(y_k - y_{k-1}) f(y_k - y_{k-1}) Q(y_k) \right).$$

Observe that the same argument of spectral representation used in the proof of (3.12) gives

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{2m} \log \sum_{y_1,\ldots,y_{2m} \in \mathbb{Z}^d} \left(\prod_{k=1}^{2m} \pi_{\alpha,\epsilon,M}(y_k - y_{k-1}) f(y_k - y_{k-1}) Q(y_k) \right) \\ &\quad = \log \rho_{\alpha,\epsilon,M}(f) \end{aligned}$$

where

$$\rho_{\alpha,\epsilon,M}(f) \equiv \sup_{|g|_2=1} \sum_{x,y\in\mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(x-y) f(x-y) \sqrt{Q\left(\frac{2\pi}{M}x\right)}\sqrt{Q\left(\frac{2\pi}{M}y\right)} g(x)g(y)$$

$$= \sup_{|g|_2=1} \sum_{x\in\mathbb{Z}^d} \pi_{\alpha,\epsilon,M}(x) f(x) \left[\sum_{y\in\mathbb{Z}^d} \sqrt{Q\left(\frac{2\pi}{M}(x+y)\right)}\sqrt{Q\left(\frac{2\pi}{M}y\right)} g(x+y)g(y)\right].$$

Hence, (3.30) follows from the fact that $\rho_{\alpha,\epsilon,M}(f) \le \sqrt{\rho_{\alpha,\epsilon,M}}$, (Cauchy–Schwartz).

By (3.27) and (3.28),

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2}\right] \le \frac{1}{2} \log\left\{\left(\frac{2\pi}{M}\right)^d \rho_{\alpha,\epsilon,M}\right\}.$$

We show in Theorem 7.1 below that

$$\limsup_{M\to\infty} \left(\frac{2\pi}{M}\right)^d \rho_{\alpha,\epsilon,M} \le \rho_{\alpha,\epsilon}.$$

This will then show that

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2}\right] \le \frac{1}{2} \log \rho_{\alpha,\epsilon}. \quad \square$$

Combining (3.14) and Lemma 3.1, we have shown that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \mathbb{E}\left[\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)^{n/2}\right] = \frac{1}{2} \log \rho_{\alpha,\epsilon}. \tag{3.31}$$

By Taylor's expansion,

$$\mathbb{E}\exp\left[\theta\left(\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)\right)^{1/2}\right] \begin{cases} <\infty & \text{for } \theta < \rho_{\alpha,\epsilon}^{-1/2} \\ =\infty & \text{for } \theta > \rho_{\alpha,\epsilon}^{-1/2}. \end{cases} \tag{3.32}$$

In addition, replacing τ by t in (3.10),

$$\eta_{\alpha,\epsilon}\left([0,t]^2\right) = \int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left|\int_0^t e^{i\lambda\cdot X_u} du\right|^2 d\lambda.$$

By the triangle inequality one has that for any $s, t > 0$

$$\left\{\int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left|\int_0^{s+t} e^{i\lambda\cdot X_u} du\right|^2 d\lambda\right\}^{1/2} \le \left\{\int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left|\int_0^{s} e^{i\lambda\cdot X_u} du\right|^2 d\lambda\right\}^{1/2} + \left\{\int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left|\int_s^{s+t} e^{i\lambda\cdot X_u} du\right|^2 d\lambda\right\}^{1/2}.$$

Notice that the random quantity

$$\int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left|\int_s^{s+t} e^{i\lambda\cdot X_u} du\right|^2 d\lambda = \int_{\mathbb{R}^d} \wp_{\alpha,\epsilon}(\lambda) \left|\int_0^{t} e^{i\lambda\cdot (X_{s+u}-X_s)} du\right|^2 d\lambda$$

is independent of $\eta_{\alpha,\epsilon}\left([0,s]^2\right)$ and has the same distribution as $\eta_{\alpha,\epsilon}\left([0,t]^2\right)$. Consequently, for any $\theta > 0$,

$$\log \mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,s+t]^2\right)\right)^{1/2}\right\}$$
$$\leq \log \mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,s]^2\right)\right)^{1/2}\right\} + \log \mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,t]^2\right)\right)^{1/2}\right\}.$$

Thus, the limit

$$L(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,t]^2\right)\right)^{1/2}\right\} \tag{3.33}$$

exists as extended real number. In view of the relation

$$\mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,\tau]^2\right)\right)^{1/2}\right\} = \int_0^\infty \mathrm{d}t \mathrm{e}^{-t} \mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,t]^2\right)\right)^{1/2}\right\},$$

and by (3.32) and the relation $\eta_{\alpha,\epsilon}\left([0,t]^2\right) = 2\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \left\{\theta \left(\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)\right)^{1/2}\right\} \begin{cases} \leq 1 & \text{for } \theta < \sqrt{\dfrac{2}{\rho_{\alpha,\epsilon}}} \\ \geq 1 & \text{for } \theta > \sqrt{\dfrac{2}{\rho_{\alpha,\epsilon}}}. \end{cases} \tag{3.34}$$

Let $(\widetilde{X}_t, \tilde{\tau})$ be an independent copy of (X_t, τ) and recall that the random measure $\zeta(\cdot)$ is defined in (2.4). The integral

$$\zeta_{\alpha,\epsilon}(A) = \iint_A \theta_{\alpha,\epsilon}(X_s - \widetilde{X}_t)\,\mathrm{d}s\,\mathrm{d}t \tag{3.35}$$

was introduced in [2] to approximate the random measure

$$\zeta(A) = \iint_A \frac{\mathrm{d}s\mathrm{d}t}{|X_s - \widetilde{X}_t|^\sigma} \quad A \subset (\mathbb{R}^+)^2$$

which is well defined whenever $0 < \sigma < \min\{2\beta, d\}$.

As in the proof of [2, Lemma 5.1], (with $p = 2$),

$$\lim_{\alpha,\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{(n!)^2} \mathbb{E}\left[\zeta\left([0,\tau]\times[0,\tilde{\tau}]\right) - \zeta_{\alpha,\epsilon}\left([0,\tau]\times[0,\tilde{\tau}]\right)\right]^n = -\infty.$$

This leads, by an similar argument, to the fact that for any $\theta > 0$

$$\lim_{\alpha,\epsilon\to 0^+} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \left\{\theta \left|\zeta\left([0,t]^2\right) - \zeta_{\alpha,\epsilon}\left([0,t]^2\right)\right|^{1/2}\right\} = 0. \tag{3.36}$$

4. Large deviations

In spite of their similarity, the large deviations in (1.17) and (1.26) require different strategies. When $0 < \sigma < \min\{\beta, d\}$, both parts, off and near the time diagonal, make contribution to the large deviation given in (1.17). When $\beta \le \sigma < \min\left\{\frac{3}{2}\beta, d\right\}$, on the other hand, renormalization makes the off-diagonal part the only source that contributes to the large deviation given in (1.26). Accordingly, different proofs are given for the two cases.

4.1. *Proof of* Theorem 1.1, $0 < \sigma < \min\{\beta, d\}$

Recall (1.31)–(1.34). Since

$$\eta^z\left([0,t]^2\right) = C^{-1}\int_{\mathbb{R}^d} \xi(t,x)\xi(t,x+z)\mathrm{d}x, \tag{4.1}$$

and translation is continuous in $L^2(\mathbb{R}^d)$, we see that $\eta^z\left([0,t]^2\right)$ is continuous in z, almost surely. Hence if $f(x) \in \mathcal{S}(R^d)$ with $\int f(x)\,\mathrm{d}x = 1$, and defining f_δ as in (3.6) we have

$$\lim_{\delta\to 0}\int\left(\iint_{[0,t]^2}|X_s - X_r - x|^{-\sigma}\,\mathrm{d}r\mathrm{d}s\right) f_\delta(x)\,\mathrm{d}x = \eta\left([0,t]^2\right). \tag{4.2}$$

But

$$\begin{aligned}
&\int\left(\iint_{[0,t]^2}|X_s - X_r - x|^{-\sigma}\,\mathrm{d}r\mathrm{d}s\right) f_\delta(x)\,\mathrm{d}x \\
&\quad = \iint_{[0,t]^2}\left(\int|X_s - X_r - x|^{-\sigma} f_\delta(x)\,\mathrm{d}x\right)\mathrm{d}r\mathrm{d}s \\
&\quad = \int |x|^{-\sigma}F_\delta(x)\,\mathrm{d}x,
\end{aligned} \tag{4.3}$$

where

$$F_\delta(x) = \iint_{[0,t]^2} f_\delta(x + X_s - X_r)\,\mathrm{d}r\mathrm{d}s \tag{4.4}$$

is in $\mathcal{S}(R^d)$ with

$$\widehat{F_\delta}(\lambda) = \iint_{[0,t]^2} \mathrm{e}^{\mathrm{i}(X_s - X_r)\cdot\lambda}\widehat{f}(\delta\,\lambda)\,\mathrm{d}r\mathrm{d}s = \widehat{f}(\delta\,\lambda)\left|\int_0^t \mathrm{e}^{\mathrm{i}\lambda\cdot X_s}\mathrm{d}s\right|^2. \tag{4.5}$$

Hence using (1.51)

$$\int |x|^{-\sigma}F_\delta(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\varphi_{d-\sigma}(\lambda)\,\widehat{f}(\delta\,\lambda)\left|\int_0^t \mathrm{e}^{\mathrm{i}\lambda\cdot X_s}\mathrm{d}s\right|^2\mathrm{d}\lambda. \tag{4.6}$$

This shows that

$$\eta\left([0,t]^2\right) = \int_{\mathbb{R}^d}\varphi_{d-\sigma}(\lambda)\left|\int_0^t \mathrm{e}^{\mathrm{i}\lambda\cdot X_s}\mathrm{d}s\right|^2\mathrm{d}\lambda. \tag{4.7}$$

Hence using (3.10)

$$\eta\left([0,t]^2\right) - \eta_{\alpha,\epsilon}\left([0,t]^2\right) = \int_{\mathbb{R}^d}\left[\varphi_{d-\sigma}(\lambda) - \wp_{\alpha,\epsilon}(\lambda)\right]\left|\int_0^t \mathrm{e}^{\mathrm{i}\lambda\cdot X_s}\mathrm{d}s\right|^2\mathrm{d}\lambda.$$

Note that $\eta\left([0,t]^2\right) - \eta_{\alpha,\epsilon}\left([0,t]^2\right) \ge 0$ by (3.5).

As in the proof of (2.3), $\left(\eta\left([0,t]^2\right)-\eta_{\alpha,\epsilon}\left([0,t]^2\right)\right)^{1/2}$ is sub-additive. Hence, for any $\theta>0$

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\eta\left([0,t]^2\right)-\eta_{\alpha,\epsilon}\left([0,t]^2\right)\right)^{1/2}\right\}\\
&\quad=\inf_{T>0}\frac{1}{T}\log\mathbb{E}\exp\left\{\theta\left(\int_{\mathbb{R}^d}\left[\varphi_{d-\sigma}(\lambda)-\wp_{\alpha,\epsilon}(\lambda)\right]\left|\int_0^T e^{i\lambda\cdot X_s}ds\right|^2 d\lambda\right)^{1/2}\right\}\\
&\quad\le\log\mathbb{E}\exp\left\{\theta\left(\int_{\mathbb{R}^d}\left[\varphi_{d-\sigma}(\lambda)-\wp_{\alpha,\epsilon}(\lambda)\right]\left|\int_0^1 e^{i\lambda\cdot X_s}ds\right|^2 d\lambda\right)^{1/2}\right\}.
\end{aligned}$$

Applying the dominated convergence theorem (based on Theorem 2.1) to the right-hand side leads to

$$\lim_{\alpha,\epsilon\to 0^+}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\eta\left([0,t]^2\right)-\eta_{\alpha,\epsilon}\left([0,t]^2\right)\right)^{1/2}\right\}=0 \tag{4.8}$$

for each $\theta>0$.

Using (3.34) and (3.15) we obtain that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta_1\left|\eta\left([0,t]^2_<\right)\right|^{1/2}\right\}\le 1,\quad \theta_1<\sqrt{\frac{2}{\rho}},\\
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta_2\left|\eta\left([0,t]^2_<\right)\right|^{1/2}\right\}\ge 1,\quad \theta_2>\sqrt{\frac{2}{\rho}}.
\end{aligned}$$

For any $\theta>0$, using the substitutions

$$t=a^{\beta/\sigma}\left(\frac{\theta}{\theta_1}\right)^{\frac{2}{2-\sigma/\beta}}\quad\text{and}\quad t=a^{\beta/\sigma}\left(\frac{\theta}{\theta_2}\right)^{\frac{2}{2-\sigma/\beta}}$$

together with the scaling (1.16) we obtain

$$\begin{aligned}
&\limsup_{a\to\infty}a^{-\beta/\sigma}\log\mathbb{E}\exp\left\{\theta a^{\beta/\sigma-1/2}\left|\eta\left([0,1]^2_<\right)\right|^{1/2}\right\}\le\left(\frac{\theta}{\theta_1}\right)^{\frac{2}{2-\sigma/\beta}},\\
&\liminf_{a\to\infty}a^{-\beta/\sigma}\log\mathbb{E}\exp\left\{\theta a^{\beta/\sigma-1/2}\left|\eta\left([0,1]^2_<\right)\right|^{1/2}\right\}\ge\left(\frac{\theta}{\theta_2}\right)^{\frac{2}{2-\sigma/\beta}}.
\end{aligned}$$

Letting $\theta_1,\theta_2\to\sqrt{2/\rho}$ gives

$$\lim_{a\to\infty}a^{-\beta/\sigma}\log\mathbb{E}\exp\left\{\theta a^{\beta/\sigma-1/2}\left(\eta\left([0,1]^2_<\right)\right)^{1/2}\right\}=\theta^{\frac{2}{2-\sigma/\beta}}(\rho/2)^{\frac{1}{2-\sigma/\beta}}. \tag{4.9}$$

Therefore, the large deviation given in (1.17) follows from the Gärtner–Ellis theorem (Theorem 2.3.6, [11]). □

4.2. *Proof of* Theorem 1.3*,* $\beta \le \sigma < \min\{\frac{3}{2}\beta, d\}$

Notice that for $a > 0$,

$$\mathbb{P}\left\{\left|\gamma\left([0,1]^2_<\right)\right| \ge a\right\} = \mathbb{P}\left\{\gamma\left([0,1]^2_<\right) \ge a\right\} + \mathbb{P}\left\{-\gamma\left([0,1]^2_<\right) \ge a\right\}. \tag{4.10}$$

We claim that

$$\lim_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{-\gamma\left([0,1]^2_<\right) \ge a\right\} = -\infty. \tag{4.11}$$

Let $m \ge 1$ be a fixed but arbitrary integer and write

$$D_m = \bigcup_{k=1}^{m-1} \left[\frac{k-1}{m}, \frac{k}{m}\right] \times \left[\frac{k}{m}, 1\right].$$

We note that

$$\begin{aligned}
\gamma\left([0,1]^2_<\right) &= \eta(D_m) - \mathbb{E}\eta(D_m) + \sum_{k=1}^m \gamma\left(\left[\frac{k-1}{m}, \frac{k}{m}\right]^2_<\right) \\
&\ge -\mathbb{E}\eta(D_m) + \sum_{k=1}^m \gamma\left(\left[\frac{k-1}{m}, \frac{k}{m}\right]^2_<\right) \\
&\overset{d}{=} -\mathbb{E}\eta(D_m) + m^{-(2-\sigma/\beta)} \sum_{k=1}^m \gamma\left([k-1,k]^2_<\right).
\end{aligned}$$

By (2.7) we see that $\mathbb{E}\eta(D_m) < \infty$ for m fixed. Thus,

$$\begin{aligned}
&\limsup_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{-\gamma\left([0,1]^2_<\right) \ge a\right\} \\
&\quad \le \limsup_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{-\sum_{k=1}^m \gamma\left([k-1,k]^2_<\right) \ge m^{2-\sigma/\beta} a\right\}.
\end{aligned}$$

Let $c > 0$ satisfy (2.10). By Chebyshev's inequality,

$$\begin{aligned}
&\mathbb{P}\left\{-\sum_{k=1}^m \gamma\left([k-1,k]^2_<\right) \ge 0m^{2-\sigma/\beta} a\right\} \\
&\quad \le \mathbb{P}\left\{\left(\sum_{k=1}^m \left|\gamma\left([k-1,k]^2_<\right)\right|\right)^{\beta/\sigma} \ge m^{2\beta/\sigma-1} a^{\beta/\sigma}\right\} \\
&\quad \le \exp\left\{-cm^{2\beta/\sigma-1} a^{\beta/\sigma}\right\} \left(\mathbb{E}\exp\left\{c\left(\sum_{k=1}^m \left|\gamma\left([k-1,k]^2_<\right)\right|\right)^{\beta/\sigma}\right\}\right) \\
&\quad \le \exp\left\{-cm^{2\beta/\sigma-1} a^{\beta/\sigma}\right\} \left(\mathbb{E}\exp\left\{c\left|\gamma\left([0,1]^2_<\right)\right|^{\beta/\sigma}\right\}\right)^m.
\end{aligned}$$

For the last inequality we used $\beta/\sigma < 1$ and the fact that the $\gamma\left([k-1,k]^2_<\right)$ are i.i.d. Therefore,

$$\limsup_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{-\gamma\left([0,1]^2_<\right) \ge a\right\} \le -cm^{2\beta/\sigma-1}.$$

Letting $m \to \infty$ gives (4.11).

By (4.10), therefore, it remains to show that

$$\lim_{a\to\infty} a^{-\beta/\sigma} \log \mathbb{P}\left\{\left|\gamma\left([0,1]^2_<\right)\right| \geq a\right\} = -2^{-\beta/\sigma}\frac{\sigma}{\beta}\left(\frac{2\beta-\sigma}{\beta}\right)^{\frac{2\beta-\sigma}{\sigma}} \rho^{-\beta/\sigma}. \tag{4.12}$$

To this end, we need the following lemma.

Lemma 4.1. *There is a $C > 0$ independent of ϵ and an $\alpha > 0$ such that for any $\theta > 0$*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\theta\left|\gamma\left([0,t]^2_<\right)\right|^{1/2}\right\} \leq C\theta^{\frac{2}{2-\sigma/\beta}} \tag{4.13}$$

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\theta\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)^{1/2}\right\} \leq C\theta^{\frac{2}{2-\sigma/\beta}}. \tag{4.14}$$

Proof. Note that since $\beta/\sigma > 2/3$, if $V \geq M^2 t^{\sigma/\beta}$ then

$$\begin{aligned} M^{-2(\beta/\sigma-1/2)} V^{\beta/\sigma} &= M^{-2(\beta/\sigma-1/2)} V^{\beta/\sigma-1/2} V^{1/2} \\ &\geq t^{1-\sigma/2\beta} V^{1/2}. \end{aligned} \tag{4.15}$$

Hence for any $M > 0$,

$$\begin{aligned} \mathbb{E}\exp\left\{\left|\gamma\left([0,t]^2_<\right)\right|^{1/2}\right\} &\leq e^{Mt} + \mathbb{E}\left(\exp\left\{\left|\gamma\left([0,t]^2_<\right)\right|^{1/2}\right\}; \left|\gamma\left([0,t]^2_<\right)\right| \geq M^2 t^2\right) \\ &= e^{Mt} + \mathbb{E}\left(\exp\left\{t^{1-\sigma/2\beta}\left|\gamma\left([0,1]^2_<\right)\right|^{1/2}\right\}; \left|\gamma\left([0,1]^2_<\right)\right| \geq M^2 t^{\sigma/\beta}\right) \\ &\leq e^{Mt} + \mathbb{E}\exp\left\{M^{-2(\beta/\sigma-1/2)}\left|\gamma\left([0,1]^2_<\right)\right|^{\beta/\sigma}\right\}. \end{aligned}$$

By Theorem 2.2, the above estimate shows that

$$C \equiv \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\left|\gamma\left([0,t]^2_<\right)\right|^{1/2}\right\} < \infty.$$

Replacing t by $\theta^{\frac{2}{2-\sigma/\beta}} t$ yields (4.13).

Observe that for any $\epsilon > 0$, $\rho_{\alpha,\epsilon} \leq \rho$. Hence by (3.34)

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\exp\left\{\lambda\left(\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)\right)^{1/2}\right\} \leq 1 \tag{4.16}$$

for any $\lambda < \sqrt{\frac{2}{\rho}}$.

On the other hand, by (3.20), for any $c > 0$ and $t > 0$,

$$\eta_{\alpha,\epsilon}\left([0,ct]^2_<\right) \stackrel{d}{=} c^{2-\sigma/\beta}\eta_{\alpha c^{(d-\sigma)/\beta},\, \epsilon c^{-1/\beta}}\left([0,t]^2_<\right). \tag{4.17}$$

Taking

$$c = \left(\frac{\theta}{\lambda}\right)^{\frac{2}{2-\sigma/\beta}}$$

and replacing t by ct, α by $\alpha c^{-(d-\sigma)/\beta}$ and ϵ by $c^{1/\beta}\epsilon$ in (4.16),

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)\right)^{1/2}\right\}\le\left(\frac{\theta}{\lambda}\right)^{\frac{2}{2-\sigma/\beta}}.$$

Letting $\lambda\to\sqrt{\frac{2}{\rho}}$ leads to

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)\right)^{1/2}\right\}\le\theta^{\frac{2}{2-\sigma/\beta}}(\rho/2)^{\frac{1}{2-\sigma/\beta}}.\quad\square$$

We now show for any $\theta>0$,

$$\lim_{\alpha,\epsilon\to0^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left|\gamma\left([0,t]^2_<\right)-\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)\right|^{1/2}\right\}=0. \tag{4.18}$$

Indeed, let the integer $N\ge1$ be large but fixed and let the sets

$$A^k_l;\quad l=0,1,\ldots,2^k-1,\ k=0,1,\ldots,N$$

be defined as in (1.23). Consider the decomposition

$$\begin{aligned}&\gamma\left([0,t]^2_<\right)-\eta_{\alpha,\epsilon}\left([0,t]^2_<\right)\\&=\sum_{l=1}^{2^{N+1}}\{\gamma-\eta_{\alpha,\epsilon}\}\left(\left[\frac{l-1}{2^{N+1}}t,\frac{l}{2^{N+1}}t\right]^2_<\right)+\sum_{k=0}^{N}\sum_{l=1}^{2^k-1}\{\gamma-\eta_{\alpha,\epsilon}\}(A^k_l).\end{aligned} \tag{4.19}$$

Notice that for each $l=0,1,\ldots,2^k-1$ and $k=0,1,\ldots,N$, and using (2.5) and an argument similar to (3.20)

$$\{\gamma-\eta_{\alpha,\epsilon}\}(A^k_l)\stackrel{d}{=}2^{-(k+1)(2-\sigma/\beta)}\left\{\zeta\left([0,t]^2\right)-\mathbb{E}\zeta\left([0,t]^2\right)-\zeta_{\bar\alpha,\bar\epsilon}\left([0,t]^2\right)\right\},$$

where $\bar\alpha=\alpha2^{-(k+1)(d-\sigma)/\beta}$, $\bar\epsilon=\epsilon2^{(k+1)/\beta}$. Also by (2.5)

$$\mathbb{E}\zeta\left([0,t]^2\right)=O\left(t^{2-\sigma/\beta}\right).$$

By (3.36), using Hölder's inequality and the fact that N is fixed, we have that

$$\lim_{\alpha,\epsilon\to0^+}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left|\sum_{k=0}^{N}\sum_{l=1}^{2^k-1}\{\gamma-\eta_{\alpha,\epsilon}\}(A^k_l)\right|^{1/2}\right\}=0. \tag{4.20}$$

Note that

$$\gamma\left(\left[\frac{l-1}{2^{N+1}}t,\frac{l}{2^{N+1}}t\right]^2_<\right)\stackrel{d}{=}\gamma\left(\left[0,\frac{t}{2^{N+1}}\right]^2_<\right).$$

Replacing t by $2^{-(N+1)}t$ in (4.13) and (4.14) we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left|\gamma\left(\left[\frac{l-1}{2^{N+1}}t,\frac{l}{2^{N+1}}t\right]^2_<\right)\right|^{1/2}\right\}\le\frac{C\theta^{\frac{2\beta}{2\beta-\sigma}}}{2^{N+1}},$$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left|\eta_{\alpha,\epsilon}\left(\left[\frac{l-1}{2^{N+1}}t,\ \frac{l}{2^{N+1}}t\right]_<^2\right)\right|^{1/2}\right\}\le\frac{C\theta^{\frac{2\beta}{2\beta-\sigma}}}{2^{N+1}},$$

for $l=1,\ldots,2^{N+1}$.

By the fact that $\frac{2\beta}{2\beta-\sigma}\ge 2$, and by Part (b), Theorem 1.2.2 in [8],

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\sum_{l=1}^{2^{N+1}}\left|\gamma\left(\left[\frac{l-1}{2^{N+1}}t,\ \frac{l}{2^{N+1}}t\right]_<^2\right)\right|\right)^{1/2}\right\}\le\frac{C\theta^{\frac{2\beta}{2\beta-\sigma}}}{2^{N+1}},$$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\left(\sum_{l=1}^{2^{N+1}}\left|\eta_{\alpha,\epsilon}\left(\left[\frac{l-1}{2^{N+1}}t,\ \frac{l}{2^{N+1}}t\right]_<^2\right)\right|\right)^{1/2}\right\}\le\frac{C\theta^{\frac{2\beta}{2\beta-\sigma}}}{2^{N+1}}.$$

Combining this with (4.19) and (4.20) leads to (4.18).

Using (3.34) and (4.18), we find as in the proof of (4.9) that for any $\theta>0$

$$\lim_{a\to\infty}a^{-\beta/\sigma}\log\mathbb{E}\exp\left\{\theta a^{\beta/\sigma-1/2}\left|\gamma\left([0,1]_<^2\right)\right|^{1/2}\right\}=\theta^{\frac{2}{2-\sigma/\beta}}(\rho/2)^{\frac{1}{2-\sigma/\beta}}.$$

Then the large deviation given in (4.12) follows from the Gärtner–Ellis theorem (Theorem 2.3.6, [11]). $\square$

5. Laws of the iterated logarithm

Proof of Theorem 1.2. Using the scaling property (1.16), our large deviation result (1.17) can be re-written as

$$\lim_{t\to\infty}(\log\log t)^{-1}\log\mathbb{P}\left\{\eta\left([0,t]_<^2\right)\ge\lambda t^{2-\sigma/\beta}(\log\log t)^{\frac{\sigma}{\beta}}\right\}$$
$$=-2^{-\beta/\sigma}\frac{\sigma}{\beta}\left(\frac{2\beta-\sigma}{\beta}\right)^{\frac{2\beta-\sigma}{\sigma}}\rho^{-\beta/\sigma}\lambda^{\beta/\sigma},\quad\lambda>0.\tag{5.1}$$

Fix $\theta>1$ and let $t_n=\theta^n$ $(n=1,2,\ldots)$. Let the fixed numbers $\lambda_1,\lambda_2>0$ satisfy

$$\lambda_1>2\rho\left(\frac{\beta}{\sigma}\right)^{\sigma/\beta}\left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{\beta}}>\lambda_2.$$

By (5.1),

$$\sum_n\mathbb{P}\left\{\eta\left([0,t_n]_<^2\right)\ge\lambda_1 t_n^{2-\sigma/\beta}(\log\log t_n)^{\frac{\sigma}{\beta}}\right\}<\infty.$$

Using the Borel–Cantelli lemma,

$$\limsup_{n\to\infty}t_n^{-(2-\sigma/\beta)}(\log\log t_n)^{-\sigma/\beta}\eta\left([0,t_n]_<^2\right)\le\lambda_1,\quad\text{a.s.}$$

Using the monotonicity of $\eta\left([0,t]_<^2\right)$ we see that for any $t_n\le t\le t_{n+1}$,

$$t^{-(2-\sigma/\beta)}(\log\log t)^{-\sigma/\beta}\eta\left([0,t]_<^2\right)$$
$$\le\frac{t_{n+1}^{2-\sigma/\beta}(\log\log t_{n+1})^{\sigma/\beta}}{t_n^{2-\sigma/\beta}(\log\log t_n)^{\sigma/\beta}}t_{n+1}^{-(2-\sigma/\beta)}(\log\log t_{n+1})^{-\sigma/\beta}\eta\left([0,t_{n+1}]_<^2\right).$$

Consequently,

$$\limsup_{t\to\infty} t^{-(2-\sigma/\beta)}(\log\log t)^{-\sigma/\beta}\eta\left([0,t]_<^2\right) \le \theta^{\frac{2\beta-\sigma}{\beta}}\lambda_1, \quad \text{a.s.}$$

Since θ can be arbitrarily close to 1 and λ_1 can be arbitrarily close to

$$2\rho\left(\frac{\beta}{\sigma}\right)^{\sigma/\beta}\left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{\beta}},$$

we have proved the upper bound for (1.20).

On the other hand, notice that the sequence

$$\eta\left([t_n,t_{n+1}]_<^2\right), \quad n=1,2,\ldots$$

is independent and for each n,

$$\eta\left([t_n,t_{n+1}]_<^2\right) \overset{d}{=} \eta\left([0,t_{n+1}-t_n]_<^2\right).$$

By (5.1), one can make θ sufficiently large so

$$\sum_n \mathbb{P}\left\{\eta\left([t_n,t_{n+1}]_<^2\right) \ge \lambda_2 t_{n+1}^{2-\sigma/\beta}(\log\log t_{n+1})^{\sigma/\beta}\right\} = \infty.$$

By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} t_{n+1}^{-(2-\sigma/\beta)}(\log\log t_{n+1})^{-\sigma/\beta}\eta\left([t_n,t_{n+1}]_<^2\right) \ge \lambda_2, \quad \text{a.s.}$$

Using the monotonicity of $\eta\left([0,t]_<^2\right)$, this leads to

$$\limsup_{t\to\infty} t^{-(2-\sigma/\beta)}(\log\log t)^{-\sigma/\beta}\eta\left([0,t]_<^2\right) \ge \lambda_2, \quad \text{a.s.}$$

Letting

$$\lambda_2 \to 2\rho\left(\frac{\beta}{\sigma}\right)^{\sigma/\beta}\left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{\beta}}$$

yields the lower bound for (1.20).

Poof of Theorem 1.4. We now turn to the proof of (1.28). With $\lambda_1, \lambda_2 > 0$ as above and using (1.26), we also have

$$\limsup_{n\to\infty} t_n^{-(2-\sigma/\beta)}(\log\log t_n)^{-\sigma/\beta}\gamma\left([0,t_n]_<^2\right) \le \lambda_1, \quad \text{a.s.} \tag{5.2}$$

for any $\theta > 1$; and

$$\limsup_{n\to\infty} t_{n+1}^{-(2-\sigma/\beta)}(\log\log t_{n+1})^{-\sigma/\beta}\gamma\left([t_n,t_{n+1}]_<^2\right) \ge \lambda_2, \quad \text{a.s.} \tag{5.3}$$

for sufficiently large θ.

However, we cannot continue as above due to the fact that $\gamma\left([0,t]_<^2\right)$ is not monotone in t. Instead, we first observe that for any $\epsilon > 0$

$$\mathbb{P}\left\{\sup_{t_n\le t\le t_{n+1}}\left|\gamma\left([0,t]_<^2\right) - \gamma\left([0,t_n]_<^2\right)\right| \ge \epsilon t_n^{2-\sigma/\beta}(\log\log t_n)^{\sigma/\beta}\right\}$$

$$= \mathbb{P}\left\{ \sup_{\theta^{-1} \le t \le 1} \left| \gamma\left([0,t]^2_<\right) - \gamma\left([0,\theta^{-1}]^2_<\right) \right| \ge \epsilon \theta^{-\frac{2\beta-\sigma}{\beta}} (\log\log t_n)^{\sigma/\beta} \right\}.$$

By Chebyshev's inequality and (2.13),

$$\sum_n \mathbb{P}\left\{ \sup_{\theta^{-1} \le t \le 1} \left| \gamma\left([0,t]^2_<\right) - \gamma\left([0,\theta^{-1}]^2_<\right) \right| \ge \epsilon \theta^{-\frac{2\beta-\sigma}{\beta}} (\log\log t_n)^{\sigma/\beta} \right\} < \infty$$

when θ is sufficiently close to 1. Consequently,

$$\limsup_{n\to\infty} t_n^{-(2-\sigma/\beta)} (\log\log t_n)^{-\sigma/\beta} \sup_{t_n \le t \le t_{n+1}} \left| \gamma\left([0,t]^2_<\right) - \gamma\left([0,t_n]^2_<\right) \right| \le \epsilon, \quad \text{a.s.} \tag{5.4}$$

Combining this with (5.2) leads to the upper bound for (2.10).

For the lower bound, observe that

$$\begin{aligned} \gamma\left([0,t_{n+1}]^2_<\right) &= \gamma\left([t_n,t_{n+1}]^2_<\right) + \gamma\left([0,t_n]^2_<\right) + \gamma\left([0,t_n] \times [t_n,t_{n+1}]\right) \\ &\ge \gamma\left([t_n,t_{n+1}]^2_<\right) + \gamma\left([0,t_n]^2_<\right) - \mathbb{E}\gamma\left([0,t_n] \times [t_n,t_{n+1}]\right). \end{aligned}$$

Given $\epsilon > 0$, an argument similar to the one used for (5.4) shows that

$$\limsup_{n\to\infty} t_{n+1}^{-(2-\sigma/\beta)} (\log\log t_{n+1})^{-\sigma/\beta} \left| \gamma\left([0,t_n]^2_<\right) \right| \le \epsilon, \quad \text{a.s.} \tag{5.5}$$

for θ sufficiently large.

In addition

$$\mathbb{E}\gamma\left([0,t_n] \times [t_n,t_{n+1}]\right) = \mathbb{E}\zeta\left([0,t_n] \times [0,t_{n+1}-t_n]\right) = O\left(t_{n+1}^{2-\sigma/\beta}\right).$$

Combining this with (5.3) and (5.5) yields

$$\limsup_{n\to\infty} t_{n+1}^{-(2-\sigma/\beta)} (\log\log t_{n+1})^{-\sigma/\beta} \gamma\left([0,t_{n+1}]^2_<\right) \ge \lambda_2 - \epsilon, \quad \text{a.s.}$$

Consequently,

$$\limsup_{t\to\infty} t^{-(2-\sigma/\beta)} (\log\log t)^{-\sigma/\beta} \gamma\left([0,t]^2_<\right) \ge \lambda_2 - \epsilon, \quad \text{a.s.}$$

This leads to the lower bound for (1.28). □

6. The stable process in a Brownian potential

Throughout this section, we take $\sigma = 2p - d$.

Proof of Corollary 1.5. From (1.36), we have that

$$F(1) \stackrel{d}{=} U\sqrt{(2C)\eta\left([0,1]^2_<\right)}$$

where U is a standard normal random variable independent of X_t. The remainder of the proof for (1.42) follows from (1.17) and a standard computation. □

Proof of Corollary 1.6. Since $\{-F(t); t \geq 0\} \overset{d}{=} \{F(t); t \geq 0\}$, we need only show that

$$\limsup_{t\to\infty} t^{-\frac{2\beta-\sigma}{2\beta}} (\log\log t)^{-\frac{\sigma+\beta}{2\beta}} F(t) = \sqrt{8C\rho}\left(\frac{\beta}{\sigma+\beta}\right)^{\frac{\sigma+\beta}{2\beta}} \left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{2\beta}}, \quad \text{a.s.} \quad (6.1)$$

Using the scaling property (1.41), our large deviation result (1.42) can be re-written as

$$\lim_{t\to\infty} (\log\log t)^{-1} \log \mathbb{P}\left\{F(t) \geq \lambda t^{\frac{2\beta-\sigma}{2\beta}} (\log\log t)^{\frac{\sigma+\beta}{2\beta}}\right\}$$

$$= -\frac{\beta+\sigma}{\beta}(8C\rho)^{-\frac{\beta}{\beta+\sigma}} \left(\frac{2\beta-\sigma}{\beta}\right)^{\frac{2\beta-\sigma}{\beta+\sigma}} \lambda^{\frac{2\beta}{\sigma+\beta}}, \quad \lambda > 0. \quad (6.2)$$

Fix $\theta > 1$ and let $t_n = \theta^n$ $(n = 1, 2, \ldots)$. Let the fixed numbers $\lambda_1, \lambda_2 > 0$ satisfy

$$\lambda_1 > \sqrt{8C\rho}\left(\frac{\beta}{\sigma+\beta}\right)^{\frac{\sigma+\beta}{2\beta}} \left(\frac{\beta}{2\beta-\sigma}\right)^{\frac{2\beta-\sigma}{2\beta}} > \lambda_2.$$

By (6.2),

$$\sum_n \mathbb{P}\left\{F(t_n) \geq \lambda_1 t_n^{\frac{2\beta-\sigma}{2\beta}} (\log\log t_n)^{\frac{\sigma+\beta}{2\beta}}\right\} < \infty \quad (6.3)$$

for any $\theta > 1$ and

$$\sum_n \mathbb{P}\left\{F(t_n) \geq \lambda_2 t_n^{\frac{2\beta-\sigma}{2\beta}} (\log\log t_n)^{\frac{\sigma+\beta}{2\beta}}\right\} = \infty \quad (6.4)$$

for $\theta > 1$ sufficiently large.

Conditioning on the stable process $\{X_t\}$, $F(t)$ is a centered Gaussian process with variance

$$\int_{\mathbb{R}^d} \xi(t,x)^2 \mathrm{d}x.$$

For $s < t$, the variance of $F(t) - F(s)$ is

$$\int_{\mathbb{R}^d} [\xi(t,x) - \xi(s,x)]^2 \, \mathrm{d}x.$$

The relation

$$\int_{\mathbb{R}^d} [\xi(t,x) - \xi(s,x)]^2 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} \xi(t,x)^2 \mathrm{d}x - \int_{\mathbb{R}^d} \xi(s,x)^2 \mathrm{d}x$$

shows that, conditionally on the stable process $\{X_t\}$, $F(t)$ is a $\mathbb{P}$-sub-additive Gaussian process [20]. By Proposition 2.2, [20],

$$\mathbb{P}\left\{\sup_{0\leq s\leq t} F(s) \geq a\right\} \leq 2\mathbb{P}\{F(t) \geq a\}, \quad a, t > 0.$$

Using (6.3) and the Borel–Cantelli lemma we see that

$$\limsup_{n\to\infty} t_n^{-(\frac{2\beta-\sigma}{2\beta})} (\log\log t_n)^{\frac{-\sigma+\beta}{2\beta}} \sup_{t\leq t_n} F(t) \leq \lambda_1, \quad \text{a.s.} \quad (6.5)$$

which leads to the desired upper bound in (6.1).

It remains to establish the lower bound.

We will say that a countable set of random variables $Y_1, Y_2, \ldots$ is associated if for any n and any coordinate-wise non-decreasing measurable functions $f, g : R^n \mapsto R$ we have

$$\mathrm{Cov}\,(f(Y_1, \ldots, Y_n), g(Y_1, \ldots, Y_n)) \geq 0, \tag{6.6}$$

and that the sequence (now the order counts) of random variables $Y_1, Y_2, \ldots$ is quasi-associated if for any $i < n$ and any coordinate-wise non-decreasing measurable functions $f : R^i \mapsto R, g : R^{n-i} \mapsto R$ we have

$$\mathrm{Cov}\,(f(Y_1, \ldots, Y_i), g(Y_{i+1}, \ldots, Y_n)) \geq 0. \tag{6.7}$$

Let $Y_x,\ x \in Z^d$ be i.i.d. standard normals. By [16, Theorem 2.1] the set $Y_x,\ x \in Z^d$ is associated. Let $h(x, y)$ be a non-negative function on $Z^d \times Z^d$ such that

$$\widetilde{h}(x.x') =: \sum_{y \in Z^d} h(x, y)h(x', y) < \infty, \tag{6.8}$$

and set $V_x = \sum_{y \in Z^d} h(x, y)Y_y$. The collection $V_x,\ x \in Z^d$ is Gaussian process with covariance

$$E(V_x V_{x'}) = f(x, x'). \tag{6.9}$$

For each $m < \infty$, let $V_{m,x} = \sum_{y \in Z^d,\, |y| \leq m} h(x, y)Y_y$. Since $V_{m,x}$ is a non-decreasing function of the $Y_y,\ |y| \leq m$, it follows from [16, (P_4)] that the set $V_{m,x},\ x \in Z^d$ is associated, and since by [16, (P_5)] association is preserved under limits, we also have that $V_x,\ x \in Z^d$ is associated.

Let now $S = \{S_0, S_1, \ldots\}$ be a random walk in Z^d. Set $g_h(m) = \sum_{i=0}^m V_{S_i}$ and $g_h(k, l) = g_h(k) - g_h(l)$. It follows from the proof of [20, Proposition 3.1] that for any n and $0 \leq s < t \leq u < v$,

$$g_h([ns], [nt]) \quad \text{and} \quad g_h([nu], [nv]) \text{ are quasi-associated.} \tag{6.10}$$

Using the stability of quasi-association under weak limits, we now show that (6.10) implies the following Lemma.

Lemma 6.1. *For any $0 \leq s < t \leq u < v$, the pair*

$$F(t) - F(s), \qquad F(v) - F(u) \tag{6.11}$$

is quasi-associated.

Proof of Lemma 6.1. Note that we can write $g_h(m) = \sum_{y \in Z^d} \sum_{i=0}^m h(S_i, y)Y_y$. Let

$$f_\epsilon(x) = \frac{e^{-\epsilon|x|}}{|x|^{\frac{\sigma+d}{2}} + \epsilon}. \tag{6.12}$$

$f_\epsilon(x)$ is a positive, continuous integrable function of x, monotone decreasing in $|x|$. We now define

$$h_{\epsilon,n,m}(x, y) = \frac{1}{n\, m^{d/2}} f_\epsilon\left(\frac{x}{n^{1/\beta}} - \frac{y}{m}\right). \tag{6.13}$$

It is clear that $h_{\epsilon,n,m}(x, y)$ satisfies (6.8). For notational convenience we set

$$g_{\epsilon,n,m}(r) = g_{h_{\epsilon,n,m}}(r). \tag{6.14}$$

We take our random walk S to be in the normal domain of attraction of X. We now show that

$$\lim_{n\to\infty} \left(g_{\epsilon,n,m}([ns],[nt]),\ g_{\epsilon,n,m}([nu],[nv])\right) \stackrel{w}{=} \left(G_{\epsilon,m}(s,t),\ G_{\epsilon,m}(u,v)\right), \tag{6.15}$$

where

$$G_{\epsilon,m}(s,t) = \frac{1}{m^{d/2}} \sum_{y\in Z^d} \left(\int_s^t f_\epsilon(X_r - y/m)\,\mathrm{d}r\right) Y_y. \tag{6.16}$$

To see this we first note that

$$E\left(\exp\left\{\mathrm{i}(a g_{\epsilon,n,m}([ns],[nt]) + b\, g_{\epsilon,n,m}([nu],[nv]))\right\}\right) = E(\mathrm{e}^{-H_{\epsilon,n,m}(a,b,s,t,u,v)/2}), \tag{6.17}$$

where

$$\begin{aligned} H_{\epsilon,n,m}(a,b,s,t,u,v) = {} & a^2 \sum_{i,j=[ns]+1}^{[nt]} \widetilde{h}_{\epsilon,m}\left(\frac{S_i}{n^{1/\beta}}, \frac{S_j}{n^{1/\beta}}\right)\frac{1}{n^2} \\ & + 2ab \sum_{i=[ns]+1}^{[nt]} \sum_{j=[nu]+1}^{[nv]} \widetilde{h}_{\epsilon,m}\left(\frac{S_i}{n^{1/\beta}}, \frac{S_j}{n^{1/\beta}}\right)\frac{1}{n^2} \\ & + b^2 \sum_{i,j=[nu]+1}^{[nv]} \widetilde{h}_{\epsilon,m}\left(\frac{S_i}{n^{1/\beta}}, \frac{S_j}{n^{1/\beta}}\right)\frac{1}{n^2} \end{aligned} \tag{6.18}$$

and

$$\widetilde{h}_{\epsilon,m}(x,x') = \frac{1}{m^d} \sum_{y\in Z^d} f_\epsilon\left(x - \frac{y}{m}\right) f_\epsilon\left(x' - \frac{y}{m}\right). \tag{6.19}$$

Similarly

$$E\left(\exp\left\{\mathrm{i}(a G_{\epsilon,m}(s,t) + b\, G_{\epsilon,m}(u,v))\right\}\right) = E(\mathrm{e}^{-H'_{\epsilon,m}(a,b,s,t,u,v)/2}), \tag{6.20}$$

where

$$\begin{aligned} H'_{\epsilon,m}(a,b,s,t,u,v) = {} & a^2 \int_{[s,t]^2} \widetilde{h}_{\epsilon,m}(X_r, X_{r'})\,\mathrm{d}r\,\mathrm{d}r' \\ & + 2ab \int_{[s,t]\times[u,v]} \widetilde{h}_{\epsilon,m}(X_r, X_{r'})\,\mathrm{d}r\,\mathrm{d}r' + b^2 \int_{[u,v]^2} \widetilde{h}_{\epsilon,m}(X_r, X_{r'})\,\mathrm{d}r\,\mathrm{d}r'. \end{aligned} \tag{6.21}$$

Thus to prove (6.15) it suffices to show that

$$\lim_{n\to\infty} H_{\epsilon,n,m}(a,b,s,t,u,v) \stackrel{w}{=} H'_{\epsilon,m}(a,b,s,t,u,v), \tag{6.22}$$

and, since $\widetilde{h}_{\epsilon,m}(x,x')$ is continuous, this follows directly from Skorohod's theorem, [29].

We next show that

$$\lim_{m\to\infty} \left(G_{\epsilon,m}(s,t),\ G_{\epsilon,m}(u,v)\right) \stackrel{w}{=} \left(G_\epsilon(s,t),\ G_\epsilon(u,v)\right), \tag{6.23}$$

where, using the notation of (1.30),

$$G_\epsilon(s,t) = \int_{R^d} \left(\int_s^t f_\epsilon(X_r - x)\,\mathrm{d}r\right) W(\mathrm{d}x). \tag{6.24}$$

But clearly

$$E\left(\exp\{(\mathrm{i}aG_\epsilon(s,t)+b\,G_\epsilon(u,v))\}\right)=E\left(\mathrm{e}^{-H'_\epsilon(a,b,s,t,u,v)/2}\right), \tag{6.25}$$

where

$$\begin{aligned}H'_\epsilon(a,b,s,t,u,v)&=a^2\int_{[s,t]^2}\widetilde{h}_\epsilon(X_r,X_{r'})\,\mathrm{d}r\,\mathrm{d}r'\\&\quad+2ab\int_{[s,t]\times[u,v]}\widetilde{h}_\epsilon(X_r,X_{r'})\,\mathrm{d}r\,\mathrm{d}r'+b^2\int_{[u,v]^2}\widetilde{h}_\epsilon(X_r,X_{r'})\,\mathrm{d}r\,\mathrm{d}r'\end{aligned} \tag{6.26}$$

and now

$$\widetilde{h}_\epsilon(x,x')=\int_{y\in R^d}f_\epsilon(x-y)f_\epsilon(x'-y)\mathrm{d}y. \tag{6.27}$$

Thus to obtain (6.23) it clearly suffices to show that

$$\lim_{m\to\infty}H'_{\epsilon,m}(a,b,s,t,u,v)\overset{L^2}{=}H'_\epsilon(a,b,s,t,u,v). \tag{6.28}$$

To see this we note first that

$$\begin{aligned}\sup_{m,x,x'}\widetilde{h}_{\epsilon,m}(x,x')&\le\sup_{m,x}\frac{1}{m^d}\sum_{y\in Z^d}\frac{\mathrm{e}^{-\epsilon|x-y/m|}}{\epsilon^2}\\&\le\sup_{m,|x|\le 1}\frac{1}{m^d}\sum_{y\in Z^d}\frac{\mathrm{e}^{-\epsilon|x-y|/m}}{\epsilon^2}\\&\le\sup_{m}\frac{1}{m^d}\sum_{y\in Z^d}\frac{\mathrm{e}^{-\epsilon|y|/m}}{\epsilon^2}\le C_\epsilon,\end{aligned} \tag{6.29}$$

and therefore (6.28) follows easily from the dominated convergence theorem.

Finally, to prove our Lemma it now suffices to show that

$$\lim_{m\to\infty}(G_\epsilon(s,t),\ G_\epsilon(u,v))\overset{w}{=}(F(t)-F(s),\ F(v)-F(u)). \tag{6.30}$$

As before, it suffices to show that

$$\lim_{\epsilon\to 0}H'_\epsilon(a,b,s,t,u,v)\overset{L^2}{=}H'(a,b,s,t,u,v), \tag{6.31}$$

where

$$\begin{aligned}H'(a,b,s,t,u,v)&=a^2\int_{[s,t]^2}\widetilde{h}(X_r,X_{r'})\,\mathrm{d}r\,\mathrm{d}r'\\&\quad+2ab\int_{[s,t]\times[u,v]}\widetilde{h}(X_r,X_{r'})\,\mathrm{d}r\,\mathrm{d}r'+b^2\int_{[u,v]^2}\widetilde{h}(X_r,X_{r'})\,\mathrm{d}r\,\mathrm{d}r'\end{aligned} \tag{6.32}$$

and

$$\widetilde{h}(x,x')=\int_{\mathbb{R}^d}|x-y|^{-\frac{\sigma+d}{2}}\,|x'-y|^{-\frac{\sigma+d}{2}}\,\mathrm{d}y=c\frac{1}{|x-x'|^\sigma}. \tag{6.33}$$

Since $\widetilde{h}_\epsilon(x, x')$ increases to $\widetilde{h}(x, x')$ as $\epsilon \to 0$, (6.31) follows easily from Theorem 2.1 and the dominated convergence theorem. $\square$

To complete the proof of the lower bound, we recall from (1.33) and (1.31) that

$$E((F(t) - F(s))(F(v) - F(u))) = c\mathbb{E} \int_s^t \int_u^v |X_a - X_b|^{-\sigma} \,\mathrm{d}a\,\mathrm{d}b$$
$$= c\mathbb{E}(|X_1|^{-\sigma}) \int_s^t \int_u^v \frac{1}{(a-b)^{\sigma/\beta}} \,\mathrm{d}a\,\mathrm{d}b. \tag{6.34}$$

It then follows as in the proof of [20, Theorem 5.2] that for any $0 < \lambda < 1$, if $0 \le s \le \lambda t < t \le u \le \lambda v < v$, then

$$\mathrm{Cov}\left(\frac{F(t) - F(s)}{(t-s)^{1-\sigma/2\beta}}, \frac{F(v) - F(u)}{(v-u)^{1-\sigma/2\beta}}\right) \le c_\lambda \left(\frac{t}{v}\right)^{\sigma/2\beta}. \tag{6.35}$$

The lower bound for our LIL then follows as in the proof of [20, Theorem 1.1]. $\square$

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Appendix. The limit as $M \to \infty$

Theorem 7.1. *Let $\rho_{\alpha,\epsilon}$ be defined in (3.13) and $\rho_{\alpha,\epsilon,M}$ be defined in (3.29). We have*

$$\limsup_{M\to\infty} \frac{M}{2\pi}^{-d} \rho_{\alpha,\epsilon,M} \le \rho_{\alpha,\epsilon}. \tag{7.1}$$

Proof. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we write $[x] = ([x_1], \ldots, [x_d])$ for the lattice part of x (We also use the notation $[\cdots]$ for parentheses without causing any confusion.) For any $f \in \mathcal{L}^2(\mathbb{Z}^d)$ with $\|f\|_2 = 1$,

$$\sum_{|x| \le (2\pi)^{-1}Ma} \wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}x\right)\left[\sum_{y\in\mathbb{Z}^d} \sqrt{Q\left(\frac{2\pi}{M}(x+y)\right)}\sqrt{Q\left(\frac{2\pi}{M}y\right)} f(x+y)f(y)\right]^2$$
$$= \int_{\{|\lambda| \le (2\pi)^{-1}Ma\}} \wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}[\lambda]\right)$$
$$\times \left[\int_{\mathbb{R}^d} \sqrt{Q\left(\frac{2\pi}{M}([\lambda]+[\gamma])\right)}\sqrt{Q\left(\frac{2\pi}{M}[\gamma]\right)} f([\lambda]+[\gamma])f([\gamma])\mathrm{d}\gamma\right]^2 \mathrm{d}\lambda$$
$$= \left(\frac{M}{2\pi}\right)^d \int_{\{|\lambda| \le a\}} \wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right)$$

$$
\times \left[\left(\frac{M}{2\pi} \right)^d \int_{\mathbb{R}^d} \sqrt{Q_M \left(\gamma + \frac{2\pi}{M} \left[\frac{M}{2\pi} \lambda \right] \right)} \sqrt{Q_M(\gamma)} \right.
$$

$$
\left. \times \; f\left(\left[\frac{M}{2\pi}\lambda \right] + \left[\frac{M}{2\pi}\gamma \right] \right) f\left(\left[\frac{M}{2\pi}\gamma \right] \right) \mathrm{d}\gamma \right]^2 \mathrm{d}\lambda, \tag{7.2}
$$

where

$$
Q_M(\lambda) = Q\left(\frac{2\pi}{M} \left[\frac{M}{2\pi}\lambda \right] \right), \quad \lambda \in \mathbb{R}^d. \tag{7.3}
$$

Write

$$
g_0(\lambda) = \left(\frac{M}{2\pi} \right)^{d/2} f\left(\left[\frac{M}{2\pi}\lambda \right] \right), \quad \lambda \in \mathbb{R}^d. \tag{7.4}
$$

We have

$$
\int_{\mathbb{R}^d} g_0^2(\lambda)\mathrm{d}\lambda = \left(\frac{M}{2\pi} \right)^d \int_{\mathbb{R}^d} f^2\left(\left[\frac{M}{2\pi}\lambda \right] \right) \mathrm{d}\lambda = \int_{\mathbb{R}^d} f^2([\lambda])\mathrm{d}\lambda = \sum_{x\in\mathbb{Z}^d} f^2(x) = 1. \tag{7.5}
$$

We can also see that under this correspondence,

$$
\left(\frac{M}{2\pi} \right)^{d/2} f\left(\left[\frac{M}{2\pi}\lambda \right] + \left[\frac{M}{2\pi}\gamma \right] \right) = g_0\left(\gamma + \frac{2\pi}{M} \left[\frac{M}{2\pi}\lambda \right] \right), \quad \lambda, \gamma \in \mathbb{R}^d. \tag{7.6}
$$

Therefore, we need only to show that for any fixed $a > 0$

$$
\limsup_{M\to\infty} \sup_{||g||_2=1} \int_{\{|\lambda|\le a\}} \wp_{\alpha,\epsilon}\left(\frac{2\pi}{M} \left[\frac{M}{2\pi}\lambda \right] \right)
$$

$$
\times \left[\int_{\mathbb{R}^d} \sqrt{Q_M\left(\gamma + \frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda \right] \right)} \sqrt{Q_M(\gamma)} g\left(\gamma + \frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda \right] \right) g(\gamma)\mathrm{d}\gamma \right]^2 \mathrm{d}\lambda
$$

$$
\le \sup_{||g||_2=1} \int_{\{|\lambda|\le a\}} \hat{h}(\epsilon\lambda)\varphi_{d-\sigma}(\lambda) \left[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)} g(\lambda+\gamma)g(\gamma)\mathrm{d}\gamma \right]^2 \mathrm{d}\lambda. \tag{7.7}
$$

To this end, note that by the inverse Fourier transformation the function

$$
U_M(\lambda) = \int_{\mathbb{R}^d} \sqrt{Q_M(\gamma+\lambda)}\sqrt{Q_M(\gamma)} g(\gamma+\lambda)g(\gamma)\mathrm{d}\gamma \tag{7.8}
$$

is the Fourier transform of the function

$$
\begin{aligned}
V_M(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} U_M(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\cdot x}\mathrm{d}\lambda \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\lambda\cdot x}\mathrm{d}\lambda \int_{\mathbb{R}^d} \sqrt{Q_M(\gamma+\lambda)}\sqrt{Q_M(\gamma)} g(\gamma+\lambda)g(\gamma)\mathrm{d}\gamma \\
&= \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}(\lambda-\gamma)\cdot x} \sqrt{Q_M(\lambda)} g(\lambda)\sqrt{Q_M(\gamma)} g(\gamma)\mathrm{d}\lambda\mathrm{d}\gamma \\
&= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\gamma} \sqrt{Q_M(\gamma)} g(\gamma)\mathrm{d}\gamma \right|^2.
\end{aligned} \tag{7.9}
$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sqrt{Q_M\left(\gamma + \frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right)} \sqrt{Q_M(\gamma)} g\left(\gamma + \frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right) g(\gamma) \mathrm{d}\gamma \\
&\quad = U_M\left(\frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right) \\
&\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left\{\mathrm{i}x \cdot \frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right\} \left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\gamma} \sqrt{Q_M(\gamma)} g(\gamma) \mathrm{d}\gamma\right|^2 \mathrm{d}x \\
&\quad \le \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|1 - \exp\left\{\mathrm{i}x \cdot \left(\lambda - \frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right)\right\}\right| \left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\gamma} \sqrt{Q_M(\gamma)} g(\gamma) \mathrm{d}\gamma\right|^2 \mathrm{d}x \\
&\qquad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\lambda} \left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\gamma} \sqrt{Q_M(\gamma)} g(\gamma) \mathrm{d}\gamma\right|^2 \mathrm{d}x. \qquad (7.10)
\end{aligned}$$

By Parseval's identity and by the fact $Q_M \le 1$,

$$\begin{aligned}
&\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\gamma} \sqrt{Q_M(\gamma)} g(\gamma) \mathrm{d}\lambda\right|^2 \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^d} Q_M(\gamma) g^2(\gamma) \mathrm{d}\gamma \le \int_{\mathbb{R}^d} g^2(\gamma) \mathrm{d}\gamma = 1. \qquad (7.11)
\end{aligned}$$

Hence, the first term on the right-hand side of (7.10) tends to 0 uniformly over $\lambda \in \mathbb{R}^d$ and over all $g \in \mathcal{L}^2(\mathbb{R}^d)$ with $||g||_2 = 1$ as $M \to \infty$. The second term on the right-hand side of (7.10) is equal to

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\lambda} V_M(x) \mathrm{d}x = U_M(\lambda) = \int_{\mathbb{R}^d} \sqrt{Q_M(\lambda+\gamma)} \sqrt{Q_M(\gamma)} g(\lambda+\gamma) g(\gamma) \mathrm{d}\gamma. \qquad (7.12)$$

Consequently, we will have (7.7) if we can prove

$$\begin{aligned}
&\limsup_{M\to\infty} \sup_{||g||_2=1} \int_{\{|\lambda|\le a\}} \wp_{\alpha,\epsilon}\left(\frac{2\pi}{M}\left[\frac{M}{2\pi}\lambda\right]\right) \\
&\qquad \times \left[\int_{\mathbb{R}^d} \sqrt{Q_M(\lambda+\gamma)} \sqrt{Q_M(\gamma)} g(\lambda+\gamma) g(\gamma) \mathrm{d}\gamma\right]^2 \mathrm{d}\lambda \\
&\quad \le \sup_{||g||_2=1} \int_{\{|\lambda|\le a\}} \wp_{\alpha,\epsilon}(\lambda) \left[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) \mathrm{d}\gamma\right]^2 \mathrm{d}\lambda. \qquad (7.13)
\end{aligned}$$

By uniform continuity of the function Q we have that $Q_M(\cdot) \to Q(\cdot)$ uniformly on $\mathbb{R}^d$. Thus, given $\epsilon > 0$ we have

$$\sup_{\lambda,\gamma\in\mathbb{R}^d} \left|\sqrt{Q_M(\lambda+\gamma)} \sqrt{Q_M(\gamma)} - \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)}\right| < \epsilon \qquad (7.14)$$

for sufficiently large M. Therefore,

$$\left\{\int_{\{|\lambda|\le a\}} \mathrm{d}\lambda \left[\int_{\mathbb{R}^d} \sqrt{Q_M(\lambda+\gamma)}\sqrt{Q_M(\gamma)}g(\lambda+\gamma)g(\gamma)\mathrm{d}\gamma\right]^2\right\}^{1/2}$$
$$\le \epsilon \left\{\int_{\{|\lambda|\le a\}} \mathrm{d}\lambda \left[\int_{\mathbb{R}^d} g(\lambda+\gamma)g(\gamma)\mathrm{d}\gamma\right]^2\right\}^{1/2}$$
$$+ \left\{\int_{\{|\lambda|\le a\}} \mathrm{d}\lambda \left[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)\mathrm{d}\gamma\right]^2\right\}^{1/2}. \tag{7.15}$$

Also, since $\|g\|_2 = 1$,

$$\int_{\{|\lambda|\le a\}} \mathrm{d}\lambda \left[\int_{\mathbb{R}^d} g(\lambda+\gamma)g(\gamma)\mathrm{d}\gamma\right]^2 \le C_d a^d, \tag{7.16}$$

where C_d is the volume of a d-dimensional unit ball. (7.13) then follows using the uniform continuity of $\wp_{\alpha,\epsilon}(\lambda)$. $\square$

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